A QUANTUM OF DIRECTION

G. A. KAVVOS

Abstract. I propose a new axiomatic approach to directed homotopy theory akin to Quillen model categories. In order to identify the appropriate classes of maps in a directed setting, I first look at categories as directed spaces.

Contents

This is a **very preliminary** draft paper, last compiled on January 16, 2019.

0.1. **Meiotopy.** Homotopy theory is predicated on the idea that there are spaces A and B which—according to our mathematical volition—should be 'the same' or 'equal,' as witnessed by some data, e.g. a (weak) homotopy equivalence. This approach has its historical roots in the fact that general topology can be 'too geometric,' and that we sometimes like to take a more coarse view. Nevertheless, since the advent of categorical homotopy theory (Riehl, 2014) and Homotopy Type Theory (HoTT) (The Univalent Foundations Program, 2013), it has developed into a mountainous industry that can be construed as a rather general *study of equality*.

In this vein, mathematicians and computer scientists alike have often considered whether a similar approach to the question of *directed homotopy* is possible. Indeed, it has been only a short decade since the first book on this subject—namely that of Grandis (2009)—appeared. The concept of directed homotopy has many formulations, and is still fuzzy. At its very basic, it pertains to a *directed* reduction of a space A to a space B through the prescription of a continuous function $f : A \to B$. In order to avoid unnecessary repetitions of the adjective 'directed,' we introduce the name *meiotopy* to stand for directed homotopy: it is a synthesis of the prefix 'meio-', which signifies reduction, and—as in the case of homotopy—the Greek word for space. Thus, meiotopy stands for the continuous reduction or deformation of one space onto another.

0.2. **Model categories.** Abstract homotopy theory takes many forms. In his thesis, Williamson (2013) remarks:

> Abstract homotopy theory has subsequently evolved in two branches, which have been explored rather independently. The first is the theory of model categories and its many variants and weakenings. The second is much less known. Its origins lie in the observation that the cylinder in topological spaces admits a much richer structure than the cylinder of the categories of simplicial or cubical sets.

Indeed, the framework of model categories—or, in more generality, that of *homotopical categories*, for which see e.g. (Riehl, 2014, §2.1)—is ubiquitous in abstract homotopy theory. The idea around which model categories is structured is that, in the context of a category $\mathcal{C}$ we may specify a class $\mathcal{W}$ of morphisms that we would like to forcefully turn into isomorphisms; we call these arrows *weak equivalences*. The process of *localisation* then constructs the *homotopy category* $\mathrm{Ho}(\mathcal{C})$, in which morphisms of $\mathcal{W}$ have formal inverses. If we think of $\mathcal{W}$ as a class of homotopy equivalences, this has the effect of identifying spaces that are homotopy equivalent. As X is equivalent to its *cylinder*—variously denoted by $I(X)$, IX or $X \times I$, even if it is not an actual product)—this also has the effect of identifying homotopic maps.

A central aim of the present paper is to adapt this type of approach to the directed setting. The goal is to formulate an axiomatic framework on 1-categories, which will provides just enough structure to define a *meiotopy category*, akin to the homotopy category that one can define when given a model category. In order to capture directed reduction, we claim that this can no longer be a 1-category. We can immediately identify two candidates as to what could take its place:

(1) The first is that of a *homotopy 2-category*, as used by Riehl and Verity (2018) in their recent draft book on ∞-categories.
(2) The second is a 1-category enriched in preorders, where $f \lesssim g$ whenever there is a meiotopy from f to g.

For the sake of simplicity, we will take the second option in this paper.

Furthermore, and if we are allowed to momentarily shoot for the moon, we would also like meiotopy structures to form some sort of presentation of ∞-categories.

0.3. **Which classes?** One comes across many challenges when attempting this adaptation. First and foremost, the notion of localisation is *inherently symmetric*, as it is a form of quotient; it is unclear what a process of *directed localisation* should be. But even before we ask that question, we come across an even more worrying problem: what is a directed analogue of the notion of weak equivalence? The lack of symmetry suggests that it should no longer be an 'equivalence,' but a 'reduction' of some sort. Indeed, the recent work of Riehl and M. Shulman (2017) on directed type theory implicitly suggests that *every* continuous function $f : A \to B$ is considered a 'reduction' from a space A to a space B. In the language of univalence, every continuous function gives rise to a directed path in the universe.

The central proposal of the present paper is that neither weak equivalence, nor arbitrary continuous functions are appropriate in the directed setting. Instead, our approach will concentrate on two notions that are inspired by those of *reflection* and *coreflection* in category theory. Studying appropriate maps in the category of small categories will naturally lead us to four classes of morphisms: *future sections*, *future retractions*, as well as their duals, *past sections*, and *past retractions*. These classes will form weak factorisation systems, and will have the left or right lifting property with respect to directed notions of fibration and cofibration.

Alas, these eight (!) classes of maps—which we will call the *one-sided structure*—will not suffice for the axiomatic development that we want. The main reason is that the factorisations corresponding to 'directed cylinders' and 'directed path spaces' no longer have their components land in any of these eight classes of maps. Instead, they belong to certain *classes of spans and cospans*. These classes will consist of directed combinations of opfibrations and fibrations, called *two-sided fibrations*; their cofibrational dual; and two classes that we will call *cyclical sections* and *cyclical retractions*, which specify ways to see a space as both a 'reflective' and 'coreflective' subspace of another space.

0.4. **Categories as directed spaces.** In order to identify, explore, and motivate the above patterns, we will extensively study factorisations of functors between *categories*. In this context, categories will be considered as *directed spaces*, or—even better—as *meiotopy 1-types*. As one can assign a *fundamental groupoid* to a space, this is analogous to the view that *groupoids* are *homotopy 1-types*.

This setting is simple enough to allow us to construct the eight classes of maps that constitute the one-sided structure (§1). Four of these classes will consist of morphisms that participate in coherent section-retraction pairs; in non-pathological cases, these pairs will be *strict (co)reflections* (adjoint section-retraction pairs where one of the two adjoint functors is full and faithful), and in quasi-pathological cases they will be something slightly less than that. The other four classes will consist of maps that have a (co)fibrational flavour. In fact, these classes will consist of directed categorifications of the notions of *Hurewicz fibrations and cofibrations*, as known from classical homotopy theory.

0.5. **Two-sided structure.** Perhaps the most unfamiliar aspect of the above discussion is the one relating to the *two-sided structure*, and its relevance in the directed setting. In this section we attempt to give some intuition, and even some rigorous results, that explain why the natural 'directed cylinder' $X \times \mathbb{2}$ and the natural 'directed path space' $X^{\mathbb{2}}$ of a small category $X \in \mathbf{Cat}$ do not fit the mould of fibrations and cofibrations, in the manner used in the theory of model categories. In fact, we will that show that they do, if and only if X is a groupoid, i.e. an undirected space (or homotopy 1-type).

0.5.1. *Directed cylinders & cofibrations.* Recall that, in a model category, a *cylinder for* X is a factorisation of the codiagonal map $\Delta_X \stackrel{\text{def}}{=} X + X \xrightarrow{[id_X, id_X]} X$ of the form

$$X + X \xrightarrow{[i_0, i_1]} I(X) \xrightarrow{w} X$$

where $[i_0, i_1]$ is a cofibration, and w is a weak equivalence. The weak factorisation system $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ always provides such a factorisation 'for free.'

Since $w \circ i_0 = w \circ i_1 = id_X$, the 2-of-3 property implies that both i_0 and i_1 are weak equivalences. We can hence picture a cylinder object for X as in Figure 1.

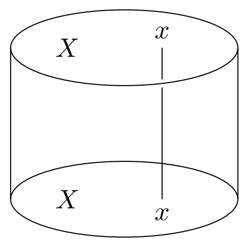


FIGURE 1. A cylinder for the space X

Thus, a cylinder $I(X)$ can be pictured as an actual cylinder whose top and bottom are separate copies of X. The two occurrences of a point $x \in X$ are connected by a vertical invertible path, which allow us to deform the top copy into the bottom, and vice versa. A homotopy $H : I(X) \to Y$ consists of two images of X in Y—the top one, and the bottom one—along with a continuous deformation between them, i.e. paths to which each line from x at the top to x at the bottom is mapped.

Suppose, now, that this cylinder is thought of as being directed downwards, i.e. as deforming the top copy of X to the bottom one, as in Figure 2.

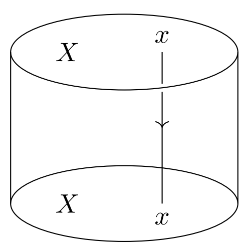


FIGURE 2. A directed cylinder (pipe) for the space X

In this case, the path from x-at-the-top to x-at-the-bottom is directed, and can only be 'walked' in the direction of gravity. We believe that it is no longer appropriate to call this a cylinder; we thus propose the name *pipe*, which is supposed to evoke a cylindrical structure along with gravity pulling any water in it towards the ground. We will usually write $\downarrow X$ to denote a pipe object for X. A *meiotopy* $K : \downarrow X \to Y$ can now be thought as a *directed deformation* of one morphism to another.

Of course, we would like this process to *not* be reversible. One of the main ideas proposed in this paper is that this non-reversibility may be captured as a condition on factorisation of the fold map. Suppose the factorisation of the codiagonal corresponding to a pipe is

$$X + X \xrightarrow{[i_0,i_1]} \downarrow X \xrightarrow{r} X$$

As $\downarrow X$ is now directed, r can no longer be seen as a weak equivalence: passing from X to $\downarrow X$ introduces an additional *quantum of direction*. Thus, the map $r : \downarrow X \to X$ can be seen as collapsing the pipe by 'erasing' this quantum. If this process follows its inherent directionality, i.e. goes 'with gravity,' then it deforms the top to bottom: it can be seen as a *future retraction*. Conversely, it can also be seen as collapsing the pipe 'anti-directionally' (in the 'anti-gravity' direction), in which case it can be seen as a *past retraction*.

It is a little bit difficult to see it right now, but it is also inconceivable to ask for $[i_0, i_1]$ to be a cofibration when our spaces are directed. Instead, it will two things: on one hand, it will be a *two-sided cofibration*, in that its components will be a cofibration and an opcofibration that are, in some sense, coherent; on the other hand, it will be a *cyclical section*, in that it will identify X as a subspace that is both 'at the start' (coreflective) and 'at the end' (reflective) of $\downarrow X$.

0.5.2. *Directed path spaces & fibrations.* The dual step to the above is also key to the model category approach to homotopy theory. This time, we look for factorisations of the diagonal $\Delta_X : X \to X \times X$ into

$$X \xrightarrow{s} PX \xrightarrow{\langle p_0,p_1\rangle} X \times X$$

with s being a weak equivalence, and $\langle p_0, p_1\rangle$ a fibration. The object PX can then be seen as a *path object for* X, and one of the factorisation systems, namely $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, provides such a factorisation 'for free.'

This situation mirrors that of fibrations in classical homotopy theory, see e.g. Arkowitz (2011, §3.3) and May (1999, §6). There, the set of paths X^I (continuous functions $I \to X$ where $I = [0, 1]$) is fibred over their endpoints: that is, the map

$$\alpha : I \to X \quad \longmapsto \quad (\alpha(0), \alpha(1)) : X \times X$$

is a fibration $X^I \xrightarrow{\langle p_0,p_1\rangle} X \times X$.

As in the case of cylinders, the factorisation in a directed case can have neither a weak equivalence, nor a fibration involved in it. A (directed) path object on X will be a factorisation

$$X \xrightarrow{s} X^{\downarrow} \xrightarrow{\langle p_0,p_1\rangle} X \times X$$

of the diagonal map $\Delta_X : X \to X \times X$.

As with cylinder objects, notice that all of s, p_0, p_1 are weak equivalences in a model category. However, the space of directed paths on X cannot possibly be weakly equivalent to X: we can certainly retract one endpoint of a path towards the other, but in a directed setting this also introduces a quantum of direction.

Consider, then, the map $s : X \to X^{\downarrow}$, which includes a point into its constant path. If we choose to retract paths towards their end point, it can be seen as picking that end point, hence as a *future section*. Conversely, if we go against the grain and retract paths backwards towards their start point, it can then be seen as picking that start point, i.e. a *past section*.

Analouglys, $\langle p_0, p_1 \rangle$ cannot possibly be seen as an ordinary fibration. Instead, it will be both a *two-sided fibration*, and also a *cyclical retraction*: it will present $X^{\downarrow}$ as an object that retracts to X in two ways, 'backwards' (towards the start of each path) and 'forwards' (towards the end of each path).

We will now substantiate these claims in the case of small categories. Given a category X, we can consider the arrow category $X^{\mathbf{2}}$: its objects are arrows

$$\begin{array}{c} x \\ \big\downarrow{\scriptstyle \alpha} \\ y \end{array}$$

which can be seen as 'paths in X.' If X is a groupoid, these paths are composable and invertible, and there is an identity (viz. stationary) path.

The morphisms of that category are commuting squares

$$\begin{array}{ccc} c & \xrightarrow{h} & x \\ {\scriptstyle \beta}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ d & \xrightarrow[k]{} & y \end{array}$$

in X. There are two functors,

$$\begin{array}{c} X^{\mathbf{2}} \\ \big\downarrow{\scriptstyle \mathrm{dom}} \\ X \end{array} \quad \text{and} \quad \begin{array}{c} X^{\mathbf{2}} \\ \big\downarrow{\scriptstyle \mathrm{cod}} \\ X \end{array}$$

that map the above square to $c \xrightarrow{h} x$ and $d \xrightarrow{k} y$ respectively. These can be combined to form the span $\langle cod, dom \rangle : X^{\mathbf{2}} \to X \times X$, and we can show that

Proposition 1. *If X is a groupoid, then $\langle cod, dom \rangle : X^{\mathbf{2}} \to X \times X$ is a fibration.*

Proof. First, notice that if X is a groupoid then both $X \times X$ and $X^{\mathbf{2}}$ are groupoids. It is also the case that, if E and B are groupoids, then $p : E \to B$ is a Grothendieck fibration precisely if each $u : b' \to p(e)$ has a lift $\breve{u} : e' \to e$ such that $p(\breve{u}) = u$.[1]

Thus, $\langle \mathrm{cod}, \mathrm{dom} \rangle : X^{\mathbf{2}} \longrightarrow X \times X$ is a fibration precisely when for each $\alpha : x \to y$, $h : c \to x$, and $k : d \to y$, there exists a dotted arrow that makes the following square commute:

$$\begin{array}{ccc} c & \xrightarrow{h} & x \\ \vdots & & \big\downarrow{\scriptstyle \alpha} \\ d & \xrightarrow[k]{} & y \end{array}$$

But, as X is a groupoid, this arrow exists, and is $k^{-1} \circ \alpha \circ h$. □

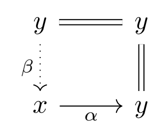
[1]More rigorously, the definitions of Grothendieck fibration and *isofibration* (there exist lifts of isomorphisms to isomorphisms) coincide in the case of groupoids; see also Proposition 11.

In cubical homotopy theory and cubical type theory, we say that the fibration computes both the 'lid' and the 'filling' (commutation) of the above 'box.'

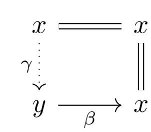
In a directed setting, the above fact is no longer true. In fact, it suffices to make the setting undirected.

Proposition 2. *If* $\langle cod, dom\rangle : X^2 \longrightarrow X \times X$ *has lifts then* X *is a groupoid. In particular, if* $\langle cod, dom\rangle$ *is a Grothendieck fibration, then* X *is a groupoid.*

Proof. Suppose $\langle \mathrm{cod}, \mathrm{dom}\rangle$ has lifts. We want to show that every arrow $\alpha : x \to y$ is invertible. We have that $(\alpha : x \to y, id_y : y \to y)$ is a morphism $(x, y) \to (y, y)$ of $X \times X$. Then $id_y : y \to y$ is an object of X^2 above (y, y), and hence there is an object of X^2, viz. an arrow $\beta : y \to x$ that makes the following diagram commute:



A similar argument gives us that there is a $\gamma : x \to y$ that makes the following diagram commute:



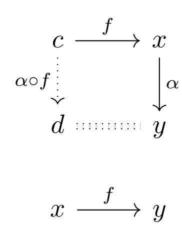
Thus β has a left inverse γ and a right inverse α. Hence $\alpha = \gamma$. □

So, if $\langle \mathrm{cod}, \mathrm{dom}\rangle : X^2 \longrightarrow X \times X$ is *not* a fibration when X is a category (i.e. a directed space), what might it be? To begin, there are some well-known facts, which can be found in e.g. Jacobs (1999) or Streicher (2018).

Proposition 3.

(1) $dom : X^2 \longrightarrow X$ *is a Grothendieck fibration*
(2) $cod : X^2 \longrightarrow X$ *is a Grothendieck opfibration.*

It might perhaps come as a surprise that the first two of these statements 'fit together.' That dom is a fibration is witnessed as follows: given an object $\alpha : x \to y \in X^2$ and a morphism $f : c \to x$ of X into its domain $\mathrm{dom}(\alpha) = x$, there is an obvious cartesian lift over f, namely



However, we need not have used the identity $id_y : y \to y$ above; in fact, any isomorphism would have worked just as well. However, the particular choice of (f, id_y) happens to be *vertical with respect to the opfibration cod.* A dual situation occurs with cartesian lifts of dom. Finally, the *doubly-indexed fibre* of X^2, namely the subcategory whose arrows are vertical with respect to *both* dom and cod is *discrete*: the only 'doubly vertical' arrows are the identities (id, id).

This kind of fibrational structure is known in the literature as a *two-sided discrete fibration*. It seems to have first appeared in the work of Street (1974, 1980) and is carefully covered in the notes by Loregian and Riehl (2018). Two-sided discrete fibrations are essentially *profunctors* seen in another light. We discuss them and their importance in directed homotopy in §2.1.

0.6. **Objectives.**

- To identify the relevant classes of maps that correspond to 'Quillen technology' (weak equivalences, fibrations, cofibrations) in directed spaces.
- To identify *factorisations* of all maps into the above that seem to govern the behaviour of directed spaces.
- To identify axioms satisfied by the above that enable us to reproduce standard arguments from the theory of model categories.
- To introduce a notion of directed localisation, or a *meiotopy category* of a category of directed spaces.

0.7. **Preliminaries on Weak Factorisation Systems.** We cover some preliminary material regarding *weak factorisation systems* (WFSs) in categories. These are included in most standard presentations of model categories, and can also be found alongside discussions of the model structure on simplicial sets, e.g. Hovey (2007), Joyal and Tierney (2008, A.2.1), or Goerss and Jardine (2009). However, a particularly lucid and comprehensive exposition can be found in the thesis of North (2017, §1).

Definition 1. A *lifting problem* in a category $\mathcal{E}$ is a commuting diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[k]{} & \cdot \end{array}$$

A (non-unique) *solution* to this lifting problem is a *diagonal filler*, i.e. an arrow d such that

$$\begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ {\scriptstyle j}\big\downarrow & \overset{d}{\nearrow} & \big\downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[k]{} & \cdot \end{array}$$

commutes. We remind the reader that diagonal fillers need not be unique.

Throughout this paper, we will use dashed lines

$$\cdot \dashrightarrow \cdot$$

to denote *unique* arrows that make a diagram commute, and dotted lines

$$\cdot \cdots\!\!\!\rightarrow \cdot$$

to denote the existence of an arrow that makes a diagram commute, but which is not necessary unique.

Definition 2. Let $\mathcal{L}$ and $\mathcal{R}$ be classes of morphisms of $\mathcal{E}$. If every lifting problem

$$\begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ {\scriptstyle j}\big\downarrow & \overset{d}{\nearrow} & \big\downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[k]{} & \cdot \end{array}$$

where $j \in \mathcal{L}$ and $f \in \mathcal{R}$ has a solution/diagonal filler d, then we write $\mathcal{L} \pitchfork \mathcal{R}$.

We say that $\mathcal{L}$ has the *left lifting property* against $\mathcal{R}$, and that $\mathcal{R}$ has the *right lifting property* against $\mathcal{L}$.

Definition 3. Let $\mathcal{L}$, $\mathcal{R}$ be classes of morphisms of a category $\mathcal{E}$.

(1) The class of morphisms that have the left lifting property against $\mathcal{R}$ is denoted by ${}^{\pitchfork}\mathcal{R}$.
(2) The class of morphisms that have the left lifting property against $\mathcal{L}$ is denoted by $\mathcal{L}^{\pitchfork}$.

This brings us to the main definition of interest.

Definition 4. A *weak factorisation system* in a category $\mathcal{E}$ is a pair $(\mathcal{A}, \mathcal{B})$ of classes of morphisms of $\mathcal{E}$ such that

(1) any $f : A \to B$ can be factorised as $f = b \circ a$ for some $a \in \mathcal{A}$ and $b \in \mathcal{B}$
(2) $(\mathcal{A}, \mathcal{B})$ are a *lifting pair*; that is, $\mathcal{A} = {}^{\pitchfork}\mathcal{B}$ and $\mathcal{A}^{\pitchfork} = \mathcal{B}$

Proposition 4. *If $(\mathcal{A}, \mathcal{B})$ are a weak factorisation system, then both $\mathcal{A}$ and $\mathcal{B}$ are closed under composition and retracts. Additionally, $\mathcal{A}$ is closed under*

(1) *pushouts*
(2) *coproducts*

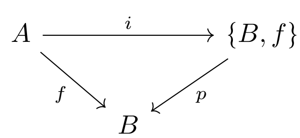
and $\mathcal{B}$ is closed under

(1) *pullbacks*
(2) *products*

Proof. As $(\mathcal{A}, \mathcal{B})$ is a lifting pair, this is (ibid., Lemma 1.1.5). □

1. Categories as Directed Spaces: I. One-sided Structure

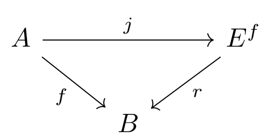
Recall that any functor $f : A \to B$ between groupoids can be factorised as



where i is injective on objects, and a categorical equivalence, and p is a Grothendieck fibration. This leads to a factorisation system of the form $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, where

$$\text{(cofibrations) } \mathcal{C} \overset{\text{def}}{=} \text{functors injective on objects}$$
$$\text{(weak equivalences) } \mathcal{W} \overset{\text{def}}{=} \text{categorical equivalences}$$
$$\text{(fibrations) } \mathcal{F} \overset{\text{def}}{=} \text{Grothendieck fibrations}$$

Similarly, we can factorise f as



where j is injective on objects (a cofibration), and r is both a categorical equivalence and a fibration. This leads to a weak factorisation system of shape $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$.

This is an essentially complete description of the model structure on groupoids; a particularly lucid presentation may be found in the notes by Joyal and Tierney (2008, §2.2).

The goal of this section is to adapt these weak factorisation systems to the meiotopic setting, namely the category of small categories **Cat**. As foreshadowed in the introduction, the symmetry provided by the class $\mathcal{W}$ disappears, and we obtain *four* different weak factorisation systems, with two versions of each of the above factorisations. Intuitively, this happens because of the extra quantum of direction that now appears, and for which there are always two choices: the forward direction, and the backward direction. Following the terminology of Grandis (2005, 2009), we will encode these choices of direction with the adjectives *future* and *past.*

We call the resulting factorisations the *North structure on* **Cat**, as their existence was shown abstractly by North (2017, §3.2.17), and one of them was covered in slightly more detail in North (2018). In private communication with the present author, North expressed the reasonable belief that these weak factorisation systems are 'folk.' Nevertheless, we are yet to find any evidence of that, and we also believe that North first noticed their importance in the context of directed type theory.

There will be no less than eight classes of maps involved in these factorisation systems. First, we have:

$$\text{(future sections) } \mathcal{FS} \stackrel{\text{def}}{=} \text{categorical future sections}$$

$$\text{(fibrations) } \mathcal{F} \stackrel{\text{def}}{=} \text{basic fibrations}$$

These will have their 'co' duals, namely

$$\text{(cofibrations) } \mathcal{C} \stackrel{\text{def}}{=} \text{basic cofibrations}$$

$$\text{(future retractions) } \mathcal{FR} \stackrel{\text{def}}{=} \text{categorical future retractions}$$

Moreover, each of the above has an 'op' dual, namely

$$\text{(past sections) } \mathcal{PS} \stackrel{\text{def}}{=} \text{categorical past sections}$$

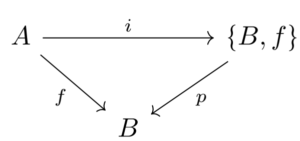
$$\text{(opfibrations) } \mathcal{F}^{\text{op}} \stackrel{\text{def}}{=} \text{basic opfibrations}$$

$$\text{(past retractions) } \mathcal{PR} \stackrel{\text{def}}{=} \text{categorical past retractions}$$

$$\text{(opcofibrations) } \mathcal{C}^{\text{op}} \stackrel{\text{def}}{=} \text{basic opcofibrations}$$

Remark 1. The 'op' superscript in the above—e.g. in the class $\mathcal{F}^{\text{op}}$ of opfibrations—is merely notation, and is *not* related to the familiar operation on categories.

1.1. **The first and second factorisations.** Let $f : A \longrightarrow B$ be a functor, this time between categories A and B. We will factorise it as



where p will be a Grothendieck fibration, and i will the right adjoint of a *strict reflection*, i.e. a full and faithful functor for which there exists $r \dashv i$ with $r \circ i = \mathsf{Id}_A$.

This factorisation is analogous to many known concepts from homotopy theory, and also closely follows that of the model structure on groupoids, as presented in Joyal and Tierney (2008, §2.2). Unlike Joyal and Tierney (ibid.), we take care to use all the information produced by their proof when the categories involved are *not* groupoids.

Moreover, since A and B are now categories, we will be able to dualise the above construction. This will immediately lead to another factorisation of f, namely

$$\begin{array}{ccc} A & \xrightarrow{\quad i' \quad} & \{f, B\} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle q'} & \\ & B & \end{array}$$

where q' will be a Grothendieck opfibration, and i' will be the left adjoint of a *strict coreflection*, i.e. a full and faithful functor such that there exists $i' \dashv r'$ for some $r' : \{f, B\} \longrightarrow A$ with $r' \circ i' = \mathsf{Id}_A$.

This factorisation was previously indistinguishable from the above. One reason was that, as we will show in §1.3, fibrations and opfibrations are indistinguishable in groupoids. Moreover, as we will see in §1.2, a (co)reflection in groupoids is a categorical equivalence, and any categorical equivalence can be strengthened to a (co)reflection.

1.1.1. *Commas & path fibrations.* The first step in constructing the above factorisations is to find analogues of standard notions of classical homotopy theory in the directed setting of **Cat**. Thankfully, it seems that the framework of *comma categories*—which is also used in describing the model structure on groupoids—already provides most of the material we need. Recall the definition of a

Definition 5 (Comma category)**.** Let $A \xrightarrow{f} X \xleftarrow{g} B$ be functors. Their *comma category* $\{f, g\}$ is the category whose objects are arrows

$$\begin{array}{c} f(a) \\ \big\downarrow{\scriptstyle \alpha} \\ g(b) \end{array}$$

in X. (More rigorously: triples $(x \in X, y \in Y, \alpha : f(x) \to g(y) \in X)$.) Its morphisms are commuting squares

$$\begin{array}{ccc} f(a) & \xrightarrow{f(\beta)} & f(a') \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha'} \\ g(b) & \xrightarrow[g(\gamma)]{} & g(b') \end{array}$$

in X. (More rigorously, tuples $(\beta : a \to a' \in A, \gamma : b \to b' \in B)$ such that the above square commutes.)

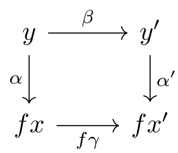
Of these, we will mostly use the cases for $B \xrightarrow{\mathsf{Id}_B} B \xleftarrow{f} A$ and $A \xrightarrow{f} B \xrightarrow{\mathsf{Id}_B} B$, which we will denote by $\{B, f\}$ and $\{f, B\}$ respectively.

It is worth describing in a little more detail what these special cases look like. The objects of the category $\{B, f\}$ are morphisms

$$\alpha : y \to fx \quad \in \quad B$$

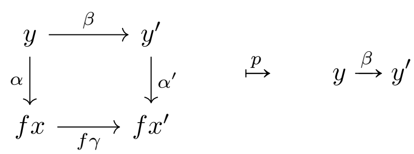
(Or, rather, triples $(y \in B, x \in A, \alpha : y \to fx)$.) The morphisms $\alpha \to \alpha'$ of this category are commuting squares

(1)



(More rigorously, they are tuples $(\beta : y \to y' \in B, \gamma : x \to x' \in A)$ such that the above diagram commutes.)

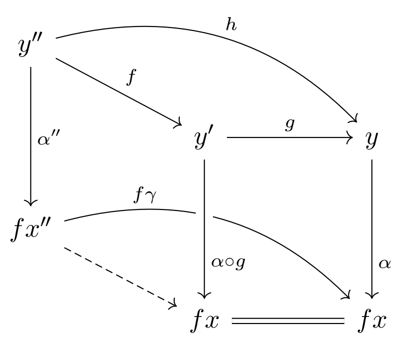
Those familiar with homotopy theory—see e.g. (Arkowitz, 2011, §3.5) and (May, 1999, §7.2)—should recognize the above as a directed categorification of the notion of *mapping path space*. In particular, if we let $p : \{B, f\} \longrightarrow B$ be the functor

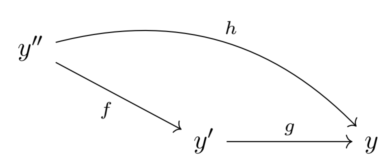


then

Proposition 5. *The functor* $p : \{B, f\} \longrightarrow B$ *is a Grothendieck fibration.*

Proof. All we ought to do is consider the following diagram, which shows that (g, id_x) is cartesian over g:





There is only one possibility for the dashed arrow, namely $f\gamma$. □

The fibre B_y of $p : \{B, f\} \longrightarrow B$ over $y \in B$ contains exactly those 'one-way paths'/morphisms $\alpha : y \to fx \in B$ whose *domain* is y. Those familiar with either classical homotopy theory, or—alternatively—homotopy type theory (The Univalent Foundations Program, 2013, Def. 4.2.5) might recognize that—were B a groupoid—the above objects would be the *homotopy fibres of* f. It is not a groupoid here, so we call them *forward meiotopy fibres* in our setting.

1.1.2. *Commas & reflections.* Consider again the case of $B \xrightarrow{\mathsf{Id}_B} B \xleftarrow{f} A$. We have shown the existence of a $p : \{B, f\} \longrightarrow B$ that is a Grothendieck fibration, and described its fibers B_y as the forward meiotopy fibres of $f : A \longrightarrow B$.

There is also another functor out of $\{B, f\}$, namely

$$q : \{B, f\} \longrightarrow A$$

which is defined by

$$\begin{array}{ccc} y & \xrightarrow{\beta} & y' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ fx & \xrightarrow[f\gamma]{} & fx' \end{array} \qquad \overset{q}{\longmapsto} \qquad x \xrightarrow{\gamma} x'$$

This functor admits a right inverse, namely

$$i : A \longrightarrow \{B, f\}$$

which is defined by taking $\gamma : x \to x' \in A$ to the 'degeneracy square'

$$\begin{array}{ccc} fx & \xrightarrow{f\gamma} & fx' \\ \| & & \| \\ fx & \xrightarrow[f\gamma]{} & fx' \end{array} \tag{2}$$

It is evident that $q \circ i = \mathsf{Id}_A$, so that i is faithful. It is also full, as a degeneracy square $\begin{array}{ccc} fx & \overset{\beta}{\dashrightarrow} & fx' \\ \| & & \| \\ fx & \xrightarrow[f\gamma]{} & fx' \end{array}$ necessarily satisfies $\beta = f\gamma$, and is hence equal to $i(\gamma)$. Moreover,

Proposition 6. *$q \dashv i$ is a strict reflection.*

Proof. The following diagram of a morphism $\alpha \to i(x')$ for $\alpha : y \to fx$ and $x' \in A$ suffices to convince us that morphisms $x = q(\alpha) \to x'$ and $\alpha \to i(x')$ naturally correspond:

$$\begin{array}{ccc} y & \xrightarrow{\beta} & fx' \\ {\scriptstyle\alpha}\downarrow & & \| \\ fx & \xrightarrow[f\gamma]{} & fx' \end{array}$$

Indeed, β is determined as $f\gamma \circ \alpha$, so the only non-trivial datum in this morphism is $\gamma : x \to x'$. □

It is worth pondering for a moment what the components of the unit $\eta : \mathsf{Id} \Rightarrow i \circ q$ are: they are arrows $\begin{array}{c} y \\ \alpha\downarrow \\ fx \end{array} \rightarrow \begin{array}{c} fx \\ \| \\ fx \end{array}$ in $\{B, f\}$; the only reasonable choice is of course

$$\eta_\alpha \stackrel{\text{def}}{=} \begin{array}{ccc} y & \xrightarrow{\alpha} & fx \\ {\scriptstyle\alpha}\downarrow & & \| \\ fx & = & fx \end{array}$$

Remark 2. Unlike the case of the model structure for groupoids, $i : A \longrightarrow \{B, f\}$ is *not* an equivalence: i might be fully faithful, but it is *not* essentially surjective. If A were a groupoid it would be, as we could recover every object $\alpha : y \to fx$ up to the isomorphism α itself, now seen as a morphism:

$$\begin{array}{ccc} y & \xrightarrow[\cong]{\alpha} & fx \\ {\scriptstyle\alpha}\downarrow & & \| \\ fx & \overset{\cong}{=} & fx \end{array}$$

Joyal and Tierney (2008, §2.2) prove this by showing that $i \dashv q$. It is easy to show that either the reflection $q \dashv i$ or the coreflection $i \dashv q$ suffice to show that i is a categorical equivalence; we will do so in §1.2.

It is now easy to see that

$$f = p \circ i$$

We can repeat the same exercise with the category $\{f, B\}$, whose objects are

$$\alpha : fx \to y$$

An analogue of Prop. 5 proves that the functor $q' : \{f, B\} \longrightarrow B$ defined by

$$\begin{array}{ccc} fx & \xrightarrow{f\beta} & fx' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ y & \xrightarrow[\gamma]{} & y' \end{array} \quad \stackrel{q'}{\longmapsto} \quad y \xrightarrow{\gamma} y'$$

is a Grothendieck opfibration. Moreover, we can define $i' : A \longrightarrow \{f, B\}$ in the same way as i, by sending $\gamma : x \to x'$ to the degeneracy square (2). A dual to Prop. 6 shows that i' is the left adjoint of a strict coreflection, and $f = q' \circ i'$.

The above two factorisations bring us very close to constructing two factorisation systems on **Cat**. Indeed, at this point we would like to show that

$$\mathcal{RR} \stackrel{\text{def}}{=} \text{right adjoints of strict reflections}$$

$$\mathcal{LC} \stackrel{\text{def}}{=} \text{left adjoints of strict coreflections}$$

$$\mathcal{GF} \stackrel{\text{def}}{=} \text{Grothendieck fibrations}$$

$$\mathcal{GF}^{\text{op}} \stackrel{\text{def}}{=} \text{Grothendieck opfibrations}$$

are closed under retraction. Unfortunately, *not one of them is.*

Hence, in order to obtain WFSs using the above factorisations, we will have to enlarge these classes of morphisms so that they are closed under retraction. This will be the topic of §§1.2–1.3: The class $\mathcal{FS}$ of *future sections* will contain all right adjoints of strict reflections, and the class $\mathcal{F}$ of *basic fibrations* will contain all Grothendieck fibrations. Similarly, the class $\mathcal{PS}$ of *past sections* will contain all left adjoints of strict coreflections, and the class $\mathcal{F}^{\mathrm{op}}$ of *basic opfibrations* will contain all Grothendieck opfibrations.

All four of these classes of morphisms will be closed under retraction (Lemma 3, Lemma 7). Furthermore, we will find that the evident duals to these classes naturally lead us to the other two weak factorisation systems of interest.

1.2. **Sections and retractions, future and past.** In the previous section we ran into a problem, namely the certain classes of maps in which useful factorisations land are not closed under retractions, and hence cannot constitute WFSs.

In this section we will solve the problem for two of these classes by weakening the definitions of *reflection* and *coreflection*. We begin by revisiting these definitions and their attendant universal properties (§1.2.1), and then proceed to weaken them (§1.2.2). These weakened forms, which always involve one fewer triangle identity than an actual adjunction, have have the required stability under retraction (§1.2.3). Finally, we show that under certain assumptions regarding idempotents in categories—which we argue are mild, intuitive, and expected in meiotopy—one can regain the missing triangle identity (§1.2.4).

1.2.1. *Reflections and coreflections.* Suppose A is a full subcategory of X. There is an *inclusion functor* $i : A \hookrightarrow X$, which is full, faithful, and injective on objects. We speak of a *(co)reflection* whenever there exists a left (or right) adjoint to it.

Definition 6 (Reflective & coreflective subcategory)**.**

(1) A full subcategory A of a category X is a *reflective subcategory of* X whenever there exists a left adjoint to the inclusion functor:

$$A \underset{i}{\overset{r}{\leftrightarrows}} X \quad (r \dashv i)$$

(2) A full subcategory A of a category X is a *coreflective subcategory of* X whenever there exists a right adjoint to the inclusion functor:

$$X \underset{r}{\overset{i}{\leftrightarrows}} A \quad (i \dashv r)$$

Since (co)reflections involve an adjunction, there is an implicit *universal property* in their definition. This is not often presented in detail. However, it is very important for us, as it provides strong spatial intuitions. First, the components

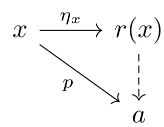
$$\eta_x : x \to r(x)$$

of the unit $\eta : \mathsf{Id}_X \Rightarrow i \circ r$ provide a *chosen path* from x to its retracted image in the reflective subcategory A. Moreover, this choice of paths is *continuous*: if we think of a morphism $p : x \to y \in X$ as a directed path in X, naturality amounts to commutation of the following square:

$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & r(x) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r(p)} \\ y & \xrightarrow[\eta_y]{} & r(y) \end{array}$$

That is: r also reflects the path p to the path $r(p) : r(x) \to r(y)$, and there is a 'homotopy' between $r(p) \circ \eta_x$ and $\eta_y \circ p$. This is a kind of continuity of r with respect to paths in X.

Second, the components of the unit are universal, which amounts to the following factorisation property: for each directed path $p : x \to a$ with $a \in A$, we have a *unique* factorisation



So each path $p : x \to a$ *into* the reflective subcategory factorises uniquely through the chosen path $\eta_x : x \to r(x)$ from the domain into its reflected image. Thus, reflective subcategories can be seen as a very orderly notion of *subspace* of a category, when that category is seen as a directed space, or meiotopy 1-type.

The counit involved in a reflective subcategory is less interesting, for reasons that are made clear by the following classic lemma of categorical homotopy theory.

Lemma 1. *If $r \dashv i$, then the following are equivalent:*

(1) *$i : A \longrightarrow X$ is fully faithful.*
(2) *the counit $\epsilon : r \circ i \overset{\cong}{\Rightarrow} \mathsf{Id}_A$ is an isomorphism.*

Dually, if $i \dashv r$, then the following are equivalent:

(1) *$i : A \longrightarrow X$ is fully faithful.*
(2) *the unit $\eta : \mathsf{Id}_X \overset{\cong}{\Rightarrow} r \circ i$ is an isomorphism.*

Proof. Due to (Gabriel and Zisman, 1967, Prop. 1.3). □

Since the inclusion $A \hookrightarrow X$ of a full subcategory is a full and faithful functor, the lemma applies to show that the counit is a natural isomorphism. This is to say that $r(a) \cong a$ for any $a \in A$. In words, if we include an object a of the subcategory into X and then reflect it to $r(a)$, we might get back something that is *not* equal to a, but merely isomorphic to it.

However, this lemma naturally leads us to an important quasi-generalisation of the notion of reflective subcategory, and our last observation also leads us to an important refinement of this generalisation. The generalisation comes first:

Definition 7 (Reflection & Coreflection)**.**

(1) A *reflection* is an adjunction $r \dashv i$ for which $i : A \to X$ is fully faithful.
(2) A *coreflection* is an adjunction $i \dashv r$ for which $i : A \to X$ is fully faithful.

That is: a (co)reflection consists of a left (or right) adjoint to a full and faithful functor i, which 'presents' A as an essentially (co)reflective subcategory of X. In fact, the nLab reports that A is equivalent to its essential image in X under i, and that the essential image is then an actual reflective subcategory. Thus, this notion is not much of a generalisation compared to reflective subcategories: the only difference is that i may collapse some objects that are distinguishable in A.

Nevertheless, this generalisation allows us to consider the following concept, which was an important theme in §1.1, and which will promptly reappear.

Definition 8 (Strict reflection & coreflection)**.**

(1) A *strict reflection* is an adjunction $r \dashv i$ for which i is full and faithful, and the components of the counit $\epsilon : r \circ i \stackrel{=}{\Rightarrow} \mathsf{Id}_A$ are identities, so that $r \circ i = \mathsf{Id}_A$.
(2) A *strict coreflection* is an adjunction $i \dashv r$ for which i is full and faithful, and the components of the unit $\eta : \mathsf{Id}_A \stackrel{=}{\Rightarrow} r \circ i$ are identities, so that $r \circ i = \mathsf{Id}_A$.

That is: a *strict reflection* is just a reflection for which the *reflector* $r : X \to A$ does not alter those objects that are already strictly in the 'subcategory,' i.e. the image of A under i. For those objects, we do not have just $r(i(a)) \cong a$, but $r(i(a)) = a$. This is reminiscent of the old American adage: "if it ain't broke, don't fix it."

Variants of this notion have appeared in the literature before. We can locate the idea occurring at least as far back as the seminal work of John Gray (1966) on fibrations: Gray would have said that r has a *rari*, that is: a *right adjoint right inverse*, namely i. (This would not have forced i to be full, but then again not all authors would take a reflective subcategory to be full.)

In both reflections and strict reflections, the adjunction provides enough data to reconstruct a universal property akin to the one we have for reflective subcategories. The components

$$\eta_x : x \longrightarrow i(r(x))$$

can be seen as directed paths from a point x to its retracted image in the reflective subcategory. Moreover, $i \circ r$ carries paths $p : x \to y$ to their image in the reflective subcategory, again in a continuous way:

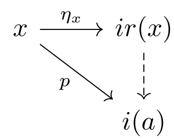
$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & i(r(x)) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle i(r(p))} \\ y & \xrightarrow[\eta_y]{} & i(r(y)) \end{array}$$

If we think of paths as being something we can walk along through time in a non-reversible manner, then a strict reflection is a kind of *future section*: it shows how we can see A as a subspace of X, in particular a subspace to which we can retract the whole of X through a natural choice of paths along each point (object of the category) given the passage of one unit of time.

Regarding the universal property, we have that for each path $p : x \to i(a)$ with an endpoint in the image of i—i.e. in the 'subcategory' A—can be factored as the chosen path followed by a path in the subcategory: there is a unique $\hat{p} : r(x) \to a$ such that

$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & ir(x) \\ & {\scriptstyle p}\searrow & \big\downarrow{\scriptstyle i(\hat{p})} \\ & & i(a) \end{array} \qquad\qquad \begin{array}{c} r(x) \\ \vdots\,{\scriptstyle \hat{p}} \\ \downarrow \\ a \end{array}$$

But we must not forget that i is full, so any path $q : ir(x) \to i(a)$ will be of the form $i(q')$ for some $q' : r(x) \to a$. Thus, there is actually a unique factorisation of any $p : x \to i(a)$ through $ir(x)$:

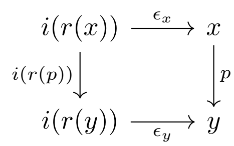


Moreover, if the reflection is *strict*, we see that $r(p) : r(x) \to a$ satisfies the universal property, so in fact $\hat{p} = r(p)$.

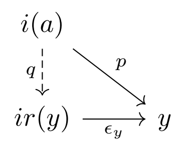
Conversely, coreflections are a kind of *past section*: the counit provides

$$\epsilon_x : i(r(x)) \longrightarrow x$$

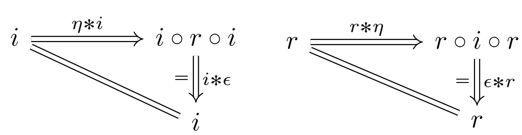
which can be read in the opposite way: if we could run time backwards, X would recede back into its coreflective subcategory A. Moreover, the naturality squares



show the relationship between paths and their image in their retracted past, before the space 'evolved' to its current state. In a manner similar to reflections, there is an induced universal property, viz. paths $p : i(a) \to y$ factor uniquely through ϵ_y:



We have not yet discussed strict (co)reflections from the viewpoint of coherence conditions—the *triangle identities*—that come with the adjunction in a reflection. These are particularly interesting in the case of a strict (co)reflection, for they are



But the counit ϵ consists of identities, and $r \circ i = \mathsf{Id}$, so these reduce to

$$\eta * i = 1_i \quad \text{and} \quad r * \eta = 1_r$$

Let us write out the first one: if a is an object of A, we have that

$$\eta_{i(a)} = id_{i(a)} : i(a) \to i(a)$$

Thus, for a given point a of the subspace, the chosen path that retracts its image $i(a)$ under the inclusion to itself is the constant path that stays put on $i(a)$ itself.

The other triangle identity is not so clear: it says that for each $x \in X$ we have

$$r\left(\eta_x : x \to i\left(r(x)\right)\right) = id_{r(x)} : r(x) \to r(x)$$

That is: if we retract the chosen path $\eta_x : x \to i(r(x))$ into the subspace A, we obtain the constant path on $r(x)$. This essentially means that we are retracting η_x along itself.

We will see in a moment that there are good reasons to be very careful in cherry-picking which coherence conditions to require, as this will is what will ensure closure under retraction. However, before we proceed we would like to consider two examples of reflection and coreflection which are of particular importance to our theory.

Consider the two categories

$$\begin{aligned} \mathbb{1} &\overset{\text{def}}{=} * \\ \mathbb{2} &\overset{\text{def}}{=} 0 \longrightarrow 1 \end{aligned}$$

The first one is the 'space of one point,' and the second one is the *directed path*, or *walking arrow*: it will play the rôle that the interval groupoid $\mathbb{I}$ plays in the homotopy theory of groupoids. There are two inclusion functors:

$$\begin{array}{c} \mathbb{1} \\ \big\downarrow {\scriptstyle i_0} \\ \mathbb{2} \end{array} \qquad \begin{array}{ccc} & * & \\ \swarrow & & \\ 0 & \longrightarrow & 1 \end{array} \qquad \begin{array}{c} \mathbb{1} \\ \big\downarrow {\scriptstyle i_1} \\ \mathbb{2} \end{array} \qquad \begin{array}{ccc} & * & \\ & & \searrow \\ 0 & \longrightarrow & 1 \end{array}$$

There is also a unique $!_{\mathbb{2}} : \mathbb{2} \longrightarrow \mathbb{1}$ in the opposite direction, and

$$!_{\mathbb{2}} \circ i_0 = \, !_{\mathbb{2}} \circ i_1 = \, !_{\mathbb{1}} = id_{\mathbb{1}}$$

The functor $!_{\mathbb{2}}$ shrinks the directed space $0 \to 1$ to a single point. We have

$$i_0 \dashv \, !_{\mathbb{2}} \dashv i_1$$

Thus, by Lemma 2, it will follow that i_1 is a future section, i_0 is a past section, and $!_{\mathbb{2}}$ is both a past retraction and a future retraction.

We have paths $0 \to i_1(!_{\mathbb{2}}(0)) = 1$ and $1 \xrightarrow{id_1} i_1(!_{\mathbb{2}}(1)) = 1$ for each object of $\mathbb{2}$, which define a natural transformation $\mathsf{Id} \Rightarrow i_1 \circ \, !_{\mathbb{2}}$, and the triangle identities hold (the second one trivially, as $\mathbb{1}$ is the terminal category). It follows that $!_{\mathbb{2}} \dashv i_1$.

In contrast, we cannot have a natural transformation $\mathsf{Id} \Rightarrow i_0 \circ \, !_{\mathbb{2}}$, as this would violate the directionality of the space $\mathbb{2}$: it would ask for a path $1 \to 0$, and no such thing exists. However, $i_0 \dashv \, !_{\mathbb{2}}$ is a strict coreflection, as there is a natural transformation $i_0 \circ \, !_{\mathbb{2}} \Rightarrow \mathsf{Id}$ that satisfies the triangle identities.

1.2.2. *A weakened form.* Right or left adjoints in a (co)reflections are particularly well-behaved in groupoids precisely because we can invert paths. For example:

Proposition 7. *A (co)reflection ($i \dashv r$) $r \dashv i$ where $i : A \longrightarrow X$ is full and faithful and X is a groupoid is an equivalence. Conversely, any equivalence can be a left or right adjoint in a (co)reflection.*

Proof. We have that the unit $\eta : \mathsf{Id}_X \Rightarrow i \circ r$ consists of isomorphisms $i(r(x)) \cong x$ for any $x \in X$, so i is fully faithful and essentially surjective. The converse is the classic result that any equivalence can be strengthened to an adjoint equivalence. □

Recall that the class of cofibrations in the model structure on groupoids consists of functors that are injective on objects, and the class of weak equivalences consists of categorical equivalences. Hence, the intersection of these two classes—which is the left class of one of the two weak factorisation systems—contains all the right adjoints of strict reflections, and all the left adjoints of strict coreflections. As mentioned above, we believe this to be an artefact of the reversibility of paths.

Thus, to obtain a structure of a directed sort, we will have to *weaken* them. Between reflection and coreflection, unit and counit, and two types of coherence, we have eight choices in total, of which only the following four are well-behaved.

Definition 9. Let $i : A \longrightarrow X$ and $r : X \longrightarrow A$ be functors, with $r \circ i = \mathsf{Id}_A$.

(1) $i : A \longrightarrow X$ is a *future section* if there exists a $\eta : \mathsf{Id}_X \Rightarrow i \circ r$ such that $\eta * i : i \Rightarrow i$ is equal to $1_i : i \Rightarrow i$. More explicitly: for each $a \in A$,

$$\eta_{i(a)} = id_{i(a)} : i(a) \to i(a)$$

(2) $i : A \longrightarrow X$ is a *past section* if there exists a $\epsilon : i \circ r \Rightarrow \mathsf{Id}_X$ such that $\epsilon * i : i \Rightarrow i$ is equal to $1_i : i \Rightarrow i$. More explicitly: for each $a \in A$,

$$\epsilon_{i(a)} = id_{i(a)} : i(a) \to i(a)$$

(3) $r : X \longrightarrow A$ is a *future retraction* if there exists a $\eta : \mathsf{Id}_X \Rightarrow i \circ r$ such that $r * \eta : r \Rightarrow r$ is equal to $1_r : r \Rightarrow r$. More explicitly: for each $x \in X$,

$$r(\eta_x) = id_{r(x)} : r(x) \to r(x)$$

(4) $r : X \longrightarrow A$ is a *past retraction* if there exists a $\epsilon : i \circ r \Rightarrow \mathsf{Id}_X$ such that $r * \epsilon : r \Rightarrow r$ is equal to $1_r : r \Rightarrow r$. More explicitly: for each $x \in X$,

$$r(\epsilon_x) = id_{r(x)} : r(x) \to r(x)$$

We will sometimes say that i is a future section *with respect to* r, if r and some $\eta : \mathsf{Id} \Rightarrow i \circ r$ witness this fact, and similarly for the rest. It is easy to see that

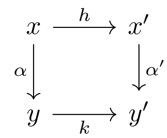
Lemma 2.

(1) *If $r \dashv i$ is a strict reflection, then i is a future section (with respect to r) and r is a future retraction (with respect to i).*
(2) *If $i \dashv r$ is a strict coreflection, then i is a past section (with respect to r) and r is a past retraction (with respect to i).*

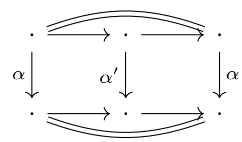
In a sense, these definitions are not new: they were implicitly discovered by North (2017, 2018), and fragments and versions of them also appear in the work of Grandis (2005, 2009). *Future homotopy equivalences* of categories, as well as their coherent variant, namely *future equivalences*, are defined in (Grandis, 2009, §3.3.1), where it is remarked that *any* reflection is a future equivalence. *Split future equivalences* are defined in (ibid., §3.3.4), which are exactly strict reflections in our sense. At that point the text refers us back to (ibid., §1.3.1), where the notion of *strong future (past) deformation retract* is defined—and it is precisely our future (past) section! In general, the theme that 'strong deformation retract needs only one coherence condition' seems to be omnipresent in Grandis' work: the same occurs in (Grandis, 2005, p. O105). Similarly, past retractions are a relaxation of the *past homotopy equivalences* of (Grandis, 2009).

1.2.3. *Closure properties.* Given any category X, we can construct its arrow category X^2, whose objects are morphisms $\begin{array}{c} x \\ \downarrow{\scriptstyle \alpha} \\ y \end{array}$ of X, and whose morphisms are pairs

(h, k) that fit in a commutative square between objects, i.e.



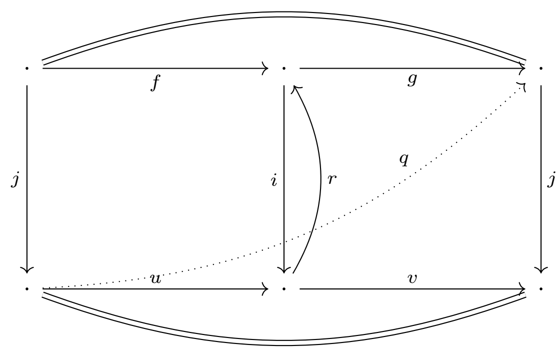
In other words, X^2 can also be described as the comma category $\{\mathsf{Id}_X, \mathsf{Id}_X\}$. An object $\alpha \in X^2$ is a *retract* of α' exactly when there is a commuting diagram



in X, where the top and bottom rows compose to identities. We can show that

Lemma 3. *The classes of future sections, past sections, future retractions, and past retractions are closed under retraction.*

Proof. We prove the first, the rest being dual.



Suppose i is a future section, and let r and η witness that. We construct

$$q \stackrel{\text{def}}{=} g \circ r \circ u$$

then

$$q \circ j = g \circ r \circ u \circ j = g \circ r \circ i \circ f = g \circ f = id$$

by commutativity of the diagram, and (i, r), (f, g) being section-retraction pairs.

It remains to construct $\eta' : \mathsf{Id} \Rightarrow j \circ q$. Given $\eta : \mathsf{Id} \Rightarrow i \circ r$, we consider

$$\eta' \stackrel{\text{def}}{=} v * \eta * u : v \circ u \Rightarrow v \circ i \circ r \circ u$$

This has the right boundary: (u, v) is a section-retraction pair, and

$$v \circ i \circ r \circ u = j \circ g \circ r \circ u = j \circ q$$

Then

$$(v * \eta * u) * j = v * (\eta * i) * f = v * 1_i * f = 1_j$$

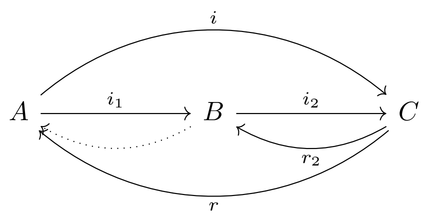
as the diagram commutes, and (f, g) is a section-retraction pair.

□

Proposition 8.

(1) *If* $i = i_2 \circ i_1$ *is a future section, and* i_2 *is any categorical section, then* i_1 *is a future section.*
(2) *If* $i = i_2 \circ i_1$ *is a past section, and* i_2 *is any categorical section, then* i_1 *is a past section.*
(3) *If* $r = r_2 \circ r_1$ *is a future retraction, and* r_1 *is any categorical retraction, then* r_2 *is a future retraction.*
(4) *If* $r = r_2 \circ r_1$ *is a past retraction, and* r_1 *is any categorical retraction, then* r_2 *is a past retraction.*

Proof. We merely show the first one, the others being very similar. Suppose



with $i = i_2 \circ i_1$, and $r_2 \circ i_2 = \mathsf{Id}$. We let $r_1 \stackrel{\text{def}}{=} r \circ i_2$. Then $r_1 \circ i_1 = r \circ i = \mathsf{Id}$. If we have a $\eta : \mathsf{Id} \Rightarrow i \circ r = i_2 \circ i_1 \circ r$ such that $\eta * i = 1$, then we consider

$$r_2 * \eta * i_2 : r_2 \circ i_2 \Rightarrow i_1 \circ r \circ i_2$$

The boundary of this can be simplified to $\mathsf{Id} \Rightarrow i_1 \circ r_1$, and of course

$$(r_2 * \eta * i_2) * i_1 = r_2 * \eta * i = 1$$

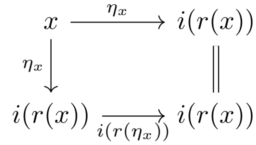
□

1.2.4. *Idempotents and Spacelike Categories.* It might seem a bit worrying to some readers that most examples of future sections $i : A \longrightarrow X$ that we consider here happen to arise as right adjoints in a strict reflections, according to Lemma 2.

Strict reflections naturally led us to future sections, as they are the weaker notion that is closed under retraction, as we showed in §1.2.3. The main idea behind that result is that we could 'transport' the unit η to the retract.

Nevertheless, our difficulty in finding a future section that is not actually part of a strict reflection begs the question: is there some reasonable and commonly satisfied condition that allows us to say that *every* future section is, in fact, a right adjoint of a strict reflection? Surprisingly, the answer is positive: so much is true when there are no non-trivial idempotents.

Recall that a future section is a $i : A \longrightarrow X$ such that there exists a $r : X \longrightarrow A$ and a $\eta : \mathsf{Id}_X \Rightarrow i \circ r$ such that $\eta * i = 1_i$. By considering the naturality square at the component η_x, we have that



commutes. The right hand side of this diagram is the component $\eta_{i(r(x))}$, which is merely the identity, as $\eta * i = 1_i$. If we apply r to this square, we obtain the equation

$$r(\eta_x) \circ r(\eta_x) = r(\eta_x)$$

which is to say that $r(\eta_x)$ is an *idempotent* in A. Recall that the missing coherence condition is $r(\eta_x) = id_{r(x)}$. Hence, we would really like this idempotent to be an identity, and we may directly ask for that:

Definition 10. A *spacelike* category is one where all idempotents are identities.

In that case,

Proposition 9. (1) *If $i : A \to X$ is a future section and A is spacelike, then i is a right adjoint in a strict reflection.*
(2) *If $i : A \to X$ is a past section and A is spacelike, then i is a left adjoint in a strict coreflection.*
(3) *If $r : X \to A$ is a future retraction and X is spacelike, then r is a left adjoint in a strict reflection.*
(4) *If $r : X \to A$ is a past retraction and X is spacelike, then r is a right adjoint in a strict coreflection.*

Proof. We have just shown the first one; the others are dual. □

One might argue that the condition that there are no non-trivial idempotents is very strong in a category. Our counterargument is based on the idea that categories are to be seen as meiotopy 1-types in this paper. In particular, if $p : x \to y$ is to be seen as a 'meiotopy class of paths,' then satisfying $p \circ p = p$ does not make much geometric sense: if walking a path twice is meiotopic to walking it once, that implies that there are no topological obstructions in 'unwinding' one of these paths, and that hence it is—up to homotopy—a trivial path. Indeed, it is a classic lemma that

Proposition 10. *Every groupoid (= homotopy 1-type) is spacelike.*

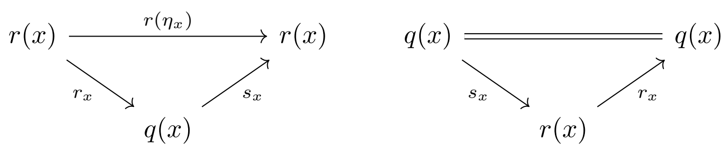
Proof. $p \circ p = p$ implies $p = id$ by cancellation. □

Nevertheless, we may still show that future sections can be 'strengthened' to be right adjoints to strict reflections even when neither of the categories are spacelike: it suffices that idempotents split.

Theorem 1. *If $i : A \longrightarrow X$ is a future section, and all idempotents in A split, then we can construct a strict reflection $q \dashv i$.*

Proof. Let $i : A \longrightarrow X$ be a future section with respect to $r : X \longrightarrow A$ and $\eta : \mathsf{Id} \Rightarrow i \circ r$, satisfying $\eta * i = 1_i$. We already know that $r(\eta_x)$ is an idempotent, so we split it to obtain a functor $q : X \longrightarrow A$ with the requisite properties.

We use the axiom of choice to pick a splitting of $r(\eta_x) : r(x) \to r(x)$ for each $x \in X$. We write



We have to exercise some care in splitting identities here: if $x = i(a)$, then $r(\eta_x) = r(\eta_{i(a)}) = r(id_{i(a)}) = id_a$, and we pick $r_{i(a)} = s_{i(a)} = id_a$.

This defines the object part of a functor $q : X \longrightarrow A$; the morphism part is

$$x \xrightarrow{f} x' \quad \longmapsto \quad q(x) \xrightarrow{s_x} r(x) \xrightarrow{r(f)} r(x') \xrightarrow{r_x} q(x')$$

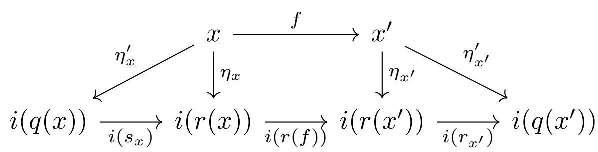
This assignment is functorial: letting $f : x \to x'$ and $g : x' \to x''$, we calculate

$$\begin{aligned} q(g) \circ q(f) &= r_{x''} \circ r(g) \circ s_{x'} \circ r_{x'} \circ r(f) \circ s_x \\ &= r_{x''} \circ r(g) \circ r(\eta_{x'}) \circ r(f) \circ s_x \\ &= r_{x''} \circ r(g) \circ r(\eta_{x'} \circ f) \circ s_x \\ &= r_{x''} \circ r(g) \circ r(ir(f) \circ \eta_x) \circ s_x \\ &= r_{x''} \circ r(g) \circ r(f) \circ r(\eta_x) \circ s_x \\ &= r_{x''} \circ r(g \circ f) \circ s_x \circ r_x \circ s_x \\ &= r_{x''} \circ r(g \circ f) \circ s_x \\ &= q(g \circ f) \end{aligned}$$

Obviously, $q(i(f : a \to a')) = r_{i(a')} \circ r(i(f)) \circ s_{i(a)} = f$, so $q \circ i = \mathsf{Id}$. It remains to define a natural transformation $\eta' : \mathsf{Id} \Rightarrow i \circ q$. We let

$$\eta'_x \stackrel{\text{def}}{=} x \xrightarrow{\eta_x} i(r(x)) \xrightarrow{i(r_x)} i(q(x))$$

Evidently, $\eta'_{i(a)} = i(r_{ia}) \circ \eta_{ia} = id_{i(a)}$. To prove naturality, consider the diagram



The bottom row is $i(q(f))$, and it commutes: the central square is a naturality square for η, the right triangle commutes by definition, and

$$i(s_x) \circ \eta'_x = i(s_x) \circ i(r_x) \circ \eta_x = i(r(\eta_x)) \circ \eta_x = \eta_x$$

where the last equality is one we showed before, by the naturality of η at η_x itself. It remains to show the 'problematic' triangle identity: we calculate

$$\begin{aligned} q(\eta'_x) &= q(i(r_x) \circ \eta_x) \\ &= r_x \circ q(\eta_x) \\ &= r_x \circ r_{i(r(x))} \circ r(\eta_x) \circ s_x \\ &= r_x \circ r(\eta_x) \circ s_x \\ &= r_x \circ s_x \circ r_x \circ s_x \\ &= id_{r(x)} \end{aligned}$$

□

The above result can be adapted to show that a past section $i : A \longrightarrow X$ can be strengthened to a left adjoint of a strict coreflection if idempotents in A split. Nevertheless, we do not currently know how to dualise this proof so that it applies to future and past retractions. This is a sign that the notion of a 'spacelike' category is particularly attractive in the meiotopic setting, and should perhaps be assumed from the outset.

1.3. **Basic fibrations and cofibrations.** The second problem that we faced when studying the factorisations in §1.1 was that the classes of Grothendieck fibrations and opfibrations did not appear to be closed under retraction, and hence could also not partake in WFSs. Indeed, this fact is corroborated by (Loregian and Riehl, 2018, Rem. 2.2.5). It follows that we must also weaken our notion of (op)fibration, so that the suitably enlarged class will be closed under retraction.

This problem did not appear in the model structure on groupoids (Joyal and Tierney, 2008, §2.2), nor in the canonical model structure on the category of small categories **Cat**, for which see the nLab, and also the note by Rezk (2000). In both cases, the class of fibrations consists of the *isofibrations*.

Definition 11. $p : E \to B$ is an isofibration whenever every isomorphism $\alpha : b' \xrightarrow{\cong} p(e)$ lifts to an isomorphism $\breve{\alpha} : e' \xrightarrow{\cong} e$ such that $p(\breve{\alpha}) = \alpha$.

In pictures:

$$\begin{array}{ccc} e' & \overset{\breve{\alpha}}{\underset{\cong}{\dashrightarrow}} & e \\ & & \\ b' & \overset{\alpha}{\underset{\cong}{\longrightarrow}} & p(e) \end{array} \qquad\qquad \begin{array}{c} E \\ \big\downarrow p \\ B \end{array}$$

We seem to have just contradicted ourselves: in §1.1 we said that the fibrations in model structure on groupoids are, in fact, the Grothendieck fibrations. The point we are trying to emphasize is that this is only a mirage precipitated by the fact that all paths in a groupoid are invertible, which makes Grothendieck fibrations and isofibrations coincide.

Proposition 11. *If E and B are groupoids, the following notions coincide:*

(1) *Grothendieck fibrations*
(2) *Grothendieck opfibrations*
(3) *isofibrations*
(4) *opisofibrations (the evident 'op' dual to the above)*

Proof. A Grothendieck fibration is an isofibration. Conversely, if p is an isofibration and B is a groupoid then any arrow $b' \xrightarrow{\cong} p(e)$ in B is an isomorphism, and hence lifts to an isomorphism $e' \xrightarrow{\cong} e$ in E. But it is easy to see that all lifts are cartesian if E is a groupoid. Similarly, Grothendieck opfibrations and opisofibrations coincide.

Finally, it is easy to see that isofibrations and opisofibrations are the same, even if B and E are not groupoids: if p is an isofibration and $p(e) \xrightarrow{\cong} b' \in B$, we can invert it to obtain $b' \xrightarrow{\cong} p(e)$, lift that to $e' \xrightarrow{\cong} e$, and then invert that to obtain $e \xrightarrow{\cong} e'$ over $p(e) \xrightarrow{\cong} b'$. □

In order to find a notion of fibration that is closed under retraction, we need to tread carefully. As before, the right idea is due to North (2018), who—in the process of describing a semantics of a directed type theory in **Cat**—discovered and sketched a factorisation system whose right class we will take to be our fibrations. This was achieved by describing these fibrations as having an *enriched lifting property*.

1.3.1. *The enriched lifting property (optional).* North (ibid.) describes a factorisation system $(L^{\to}, R^{\to})$ on **Cat**, which is shown to contain all Grothendieck opfibrations in $R^{\to}$. Only one clue about the nature of this factorisation system is given. After inverting it to rid ourselves of the 'op,' the clue is that the desired right class consists of the functors that have the *enriched right lifting property* with respect to the inclusion

$$\begin{array}{c} \mathbb{1} \\ \big\downarrow{\scriptstyle i_1} \\ \mathbb{2} \end{array} \qquad \begin{array}{ccc} & * & \\ & & \searrow \\ 0 & \longrightarrow & 1 \end{array}$$

But what is this mysterious *enriched* right lifting property? The key lies in viewing factorisation systems through hom-sets. As we mentioned before in §0.7, a solution to a lifting problem $\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$ is a diagonal filler that makes both triangles commute:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

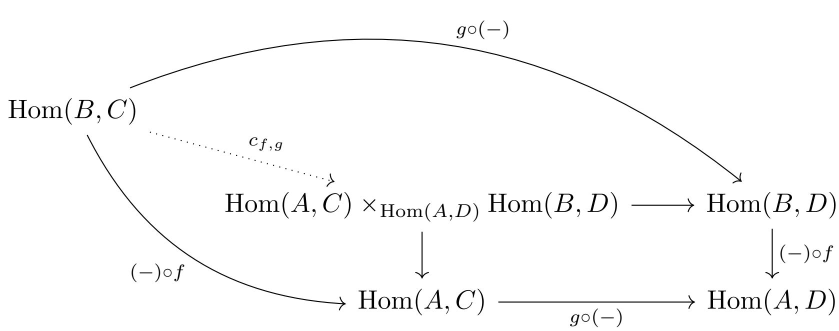
Suppose we now 'fix' $f : A \to B$ and $g : C \to D$. Instances of the above problem are arrows h and k such that the above square commutes. These instances can be understood to be elements of the following pullback:

$$\begin{array}{ccc} \mathrm{Hom}(A,C) \times_{\mathrm{Hom}(A,D)} \mathrm{Hom}(B,D) & \longrightarrow & \mathrm{Hom}(B,D) \\ \big\downarrow & & \big\downarrow{\scriptstyle (-)\circ f} \\ \mathrm{Hom}(A,C) & \xrightarrow[g\circ(-)]{} & \mathrm{Hom}(A,D) \end{array}$$

Given any filler $d : B \to C$, we can construct such a 'problem instance' to which d itself is a solution, namely

$$\begin{array}{ccc} A & \xrightarrow{d\circ f} & C \\ {\scriptstyle f}\big\downarrow & \overset{d}{\nearrow} & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[g\circ d]{} & D \end{array}$$

This means that the following diagram commutes:



We thus get a canonical *comparison map*

$$c_{f,g} : \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C) \times_{\mathrm{Hom}(A,D)} \mathrm{Hom}(B,D)$$

It then follows that there exist solutions to lifting problems if we can construct a section to this comparison map:

$$\mathrm{Hom}(B,C) \xrightarrow{c_{f,g}} \mathrm{Hom}(A,C) \times_{\mathrm{Hom}(A,D)} \mathrm{Hom}(B,D), \qquad s : \mathrm{Hom}(A,C) \times_{\mathrm{Hom}(A,D)} \mathrm{Hom}(B,D) \dashrightarrow \mathrm{Hom}(B,C)$$

When given an instance (h,k) of a lifting problem, s gives us a filler that solves the particular problem. Then f and g have the *enriched lifting property* with respect to each other. A more general version of this idea is presented in (Riehl, 2014, §13): the name *enriched* comes from the fact that the above hom-objects need not be sets, but could live in any symmetric monoidal category $\mathcal{V}$, as per the usual practice of enriched category theory.

The same trick is used to great effect in a (locally) cartesian closed setting (which is enrichment of a category over itself) by Awodey (2018) within the context of natural models of homotopy type theory. The following argument, which may be found as (Riehl, 2014, Exercise 11.1.9) or (Awodey, 2018, Lemma 2.15) characterizes the exact lifting property that we have obtained.

Lemma 4. *Let $f : A \to B$ and $g : C \to D$ be morphisms in a category with products. Then the following are equivalent:*

(1) *A section to the comparison map $c_{f,g} : Hom(A,C) \times_{Hom(A,D)} Hom(B,D) \to Hom(B,C)$.*
(2) *Solutions to the filling problems of the following form, for any X:*

$$\begin{array}{ccc} X \times A & \xrightarrow{h} & C \\ {\scriptstyle id_X \times f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ X \times B & \xrightarrow[k]{} & D \end{array}$$

which are natural *in X, in the sense that for any $c : Y \to X$, the following diagram commutes:*

$$\begin{array}{ccccc} Y \times A & \xrightarrow{c \times id_A} & X \times A & \xrightarrow{h} & C \\ {\scriptstyle id_Y \times f}\downarrow & d' & \downarrow{\scriptstyle id_X \times f} & d & \downarrow{\scriptstyle g} \\ Y \times B & \xrightarrow[c \times id_B]{} & X \times B & \xrightarrow[k]{} & D \end{array}$$

i.e. writing $d[h,k]$ for d and $d\left[h \circ (c \times id_A), k \circ (c \times id_B)\right]$ for d',

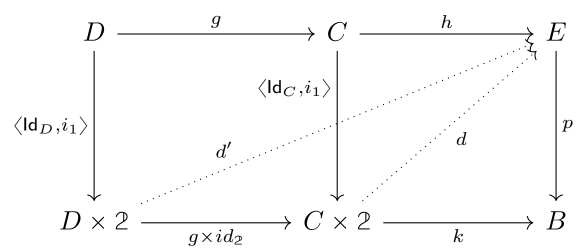
$$d[a,b] \circ (c \times id_B) = d\left[a \circ (c \times id_A), b \circ (c \times id_B)\right]$$

Having demystified the enriched right lifting property, we now consider which functors satisfy it. Suppose, then, that $p : E \longrightarrow B$ has the enriched right lifting property with respect to the inclusion $i_1 : \mathbb{1} \longrightarrow \mathbb{2}$ in the codomain of the single arrow of $\mathbb{2} = 0 \to 1$. Then, according to the above discussion, we can fill the square

$$\begin{array}{ccc} C & \xrightarrow{h} & E \\ {\scriptstyle \langle \mathsf{Id}_C, i_1 \rangle}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle p} \\ C \times \mathbb{2} & \xrightarrow[k]{} & B \end{array}$$

where we have surreptiously replaced $C \times \mathbb{1}$ with C. Staring at this for a moment is enough to convince one that k is really a natural transformation $\sigma : k_0 \Rightarrow k_1$ from the functors $k_0 = k(-, 0), k_1 = k(-, 1) : C \to B$, and f is over k_1. A diagonal filler d then is a natural transformation $\breve{\sigma} : d_0 \Rightarrow d_1$, where $d_0 = d(-, 0), d_1 = d(-, 1) : C \to E$. Commutation of the bottom triangle yields that d_0 and d_1 are respectively over b_0 and b_1, and commutation of the top triangle gives that $d_1 = h$.

We are also given that these fillers are natural, in the sense that for any functor $g : D \longrightarrow C$, there is a filler of the following outer square that makes the whole diagram commute:



If such fillers exist for all such b and f, what we obtain is a directed categorification of the notion of *Hurewicz fibrations*, which are continuous functions that satisfy the *homotopy lifting property*, in that one can lift homotopies from their base to the total space. This is the impetus behind our definition of *basic fibration*.

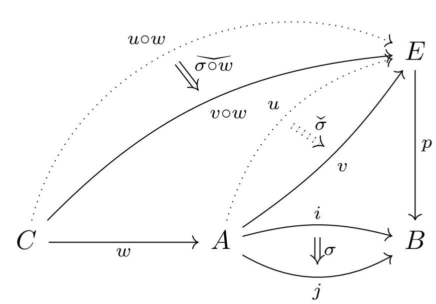
1.3.2. *Fibrations and cofibrations.* We will consider a notion of fibrations that is as general as possible: a fibration will be a functor that satisfies a general 'lifting property.' This idea has appeared before, e.g. in (Pavlović, 1990, §II.1.7).

Definition 12 (Lifting)**.** Let $p : E \longrightarrow B$, and let $\sigma : i \Rightarrow j$ be a natural transformation for functors $i, j : A \longrightarrow B$. Let $v : A \longrightarrow E$ be *over* j, in the sense that $p \circ v = j$. A *lifting* of σ along p consists of a functor $u : A \longrightarrow E$ over i, and a $\breve{\sigma} : u \Rightarrow v$ over σ, in the sense that $p * \breve{\sigma} = \sigma$.

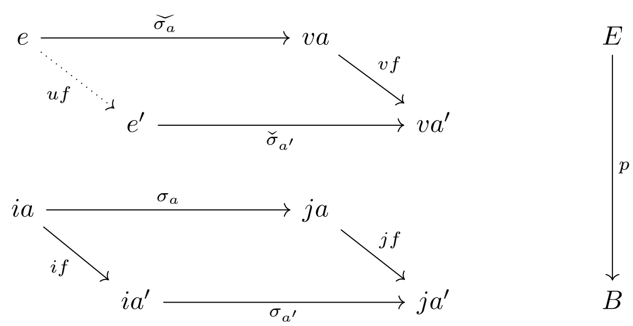
Definition 13 (Basic fibration)**.** A *basic fibration* is a functor $p : E \longrightarrow B$ along which liftings of any $\sigma : i \Rightarrow j$ and $v : A \longrightarrow B$ over j exist. Moreover, these liftings are required to be

(1) *natural*, in that for $w : C \to B$ we have $\widetilde{\sigma * w} = \breve{\sigma} * w$, and
(2) *normalised*, in that $\breve{1_j} = 1_v$.

In pictures:



This corresponds to the following lifting property, for each $f : a \to a' \in A$:



The natural transformation σ provides the bottom square, and vf is an arrow above jf. The fibration then gives both the lifts $\breve{\sigma}_a$, $\breve{\sigma}_{a'}$, as well as uf above if. Moreover, the uf's are chosen so as to make u a a functor, and hence $\breve{\sigma}$ a natural transformation.

Grothendieck fibrations satisfy this kind of lifting property. The following lemma is given in (ibid., §II.1.7), and is a simplified version of a much stronger result which can be found in (Gray, 1966, Theorem 2.10).

Lemma 5 (Gray)**.** *A functor $p : E \longrightarrow B$ is a (cloven) Grothendieck fibration iff every $\sigma : i \Rightarrow j$ and $v : A \longrightarrow E$ with $p \circ v = j$ has a lifting $\breve{\sigma}$ along p which is* cartesian*, in the sense that every component of $\breve{\sigma}$ is a cartesian morphism in E with respect to p.*

Corollary 1. *Every Grothendieck fibration is a basic fibration.*

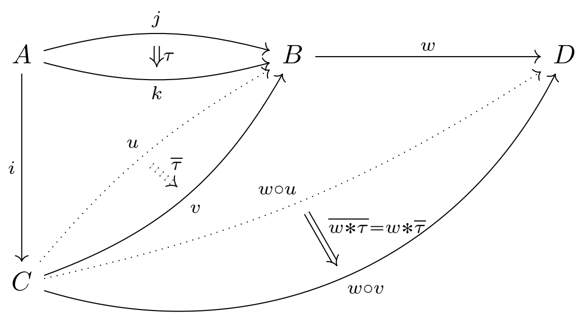
A very attractive aspect of this formulation of a fibration through a lifting property is that one immediately obtains duals. The 'co'-dual notion that suggests itself is exactly what one expects, namely a categorical notion of *cofibration*. Instead of lifting, the underlying concept is now that of *extension*:

Definition 14 (Extension)**.** Let $i : A \longrightarrow C$, and let $\tau : j \Rightarrow k$ be a natural transformation for functors $j, k : A \longrightarrow B$. Let $v : C \longrightarrow B$ be *under* k, in the sense that $v \circ i = k$. An *extension* of τ along i consists of a functor $u : C \longrightarrow B$ under j, and a $\overline{\tau} : u \Rightarrow v$ under τ, in the sense that $\overline{\tau} * i = \tau$.

Definition 15 (Basic cofibration)**.** A *basic cofibration* is a functor $i : A \longrightarrow C$ along which extensions of any $\tau : j \Rightarrow k$ and $v : C \longrightarrow B$ under k exist. Moreover, these extensions are required to be

(1) *natural*, in that for $w : B \to D$ we have $\overline{w * \tau} = w * \overline{\tau}$, and
(2) *normalised*, in that $\overline{1_k} = 1_v$ whenever $\tau_c = id$.

In pictures:



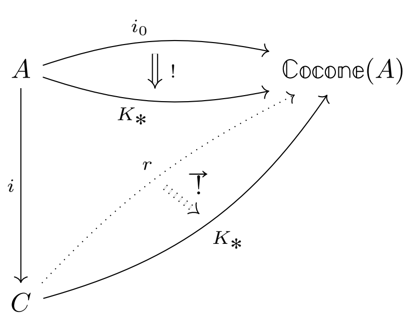
Whereas all sorts of fibrations abound in category theory, cofibrations are not so commonly encountered. A notable exception is the seminal article of Gray (1966), where a notion of *Grothendieck cofibration* makes a short appearance. Nevertheless, cofibrations are ubiquitous in topology and classical homotopy theory. There, the continuous maps that have a *homotopy extension property* are referred to as *Hurewicz cofibrations*: see Arkowitz (2011, §3.2) or May (1999, §6).

It is natural to ask if, like in classical homotopy theory, all basic cofibrations are, in one way or another, 'inclusions.' In fact, a well-known exercise in classical homotopy theory—which appears in most textbooks, see e.g. Arkowitz (2011, Prop. 3.2.6)—can be adapted to show that all basic cofibrations are *embeddings*.

Lemma 6. *Any basic (op)cofibration $i : A \longrightarrow C$ is an* embedding*, i.e. a faithful functor that is injective on objects.*[2]

Proof. We only show the case for cofibrations, the op being dual. Suppose $i : A \longrightarrow C$ is a basic cofibration. Define the category $\mathbb{Cocone}(A)$ to consist of A with an 'extra' terminal object $*$ added, along with a unique arrow $!_a : a \to *$ from every object $a \in A$. Let $i_0 : A \longrightarrow \mathbb{Cocone}(A)$ be the inclusion, and let $K_* : A \longrightarrow \mathbb{Cocone}(A)$ be the functor that collapses all of A to $*$.

The unique arrows to $*$ evidently form a natural transformation $! : i_0 \Rightarrow K_*$. Moreover, K_* can be trivially extended to all of C. Thus, we can extend $!$ along i to obtain a natural transformation $\overrightarrow{!} : r \Rightarrow K_*$:



Thus we obtain $r : C \longrightarrow \mathbb{Cocone}(A)$ such that $r \circ i = i_0$, so i is faithful and injective on objects, as $i_0(f : a \to a') = f : a \to a'$. □

[2]An embedding that is often called a *full embedding*: see the nLab

Producing non-trivial illustrative examples of basic cofibrations by hand is not a straightforward exercise. Nevertheless, in §2.4 we shall see a way to reliably produce very many examples from a stronger kind of contraption, namely a *profunctor*.

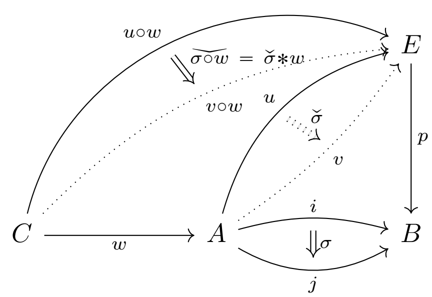
1.3.3. *Basic opfibrations and opcofibrations.* Taking the 'op' dual of the above notions leads to another two notions of (co)fibration, namely *opfibrations* and *opcofibrations*. The definitions are spelled out for completeness.

Definition 16 (Oplifting)**.** Let $p : E \longrightarrow B$, and let $\sigma : i \Rightarrow j$ be a natural transformation for functors $i, j : A \longrightarrow B$. Let $u : A \longrightarrow E$ be *over* i, in the sense that $p \circ u = i$. An *oplifting* of σ along p consists of a functor $v : A \longrightarrow E$ over j, and a $\breve{\sigma} : u \Rightarrow v$ over σ, in the sense that $p * \breve{\sigma} = \sigma$.

Definition 17 (Basic opfibration)**.** A *basic opfibration* is a functor $p : E \longrightarrow B$ along which opliftings of any $\sigma : i \Rightarrow j$ and $u : A \longrightarrow B$ over i exist. Moreover, these liftings are required to be

(1) *natural*, in that for $w : C \to B$ we have $\widetilde{\sigma * w} = \breve{\sigma} * w$, and
(2) *normalised*, in that $\breve{1}_i = 1_u$.

In pictures:

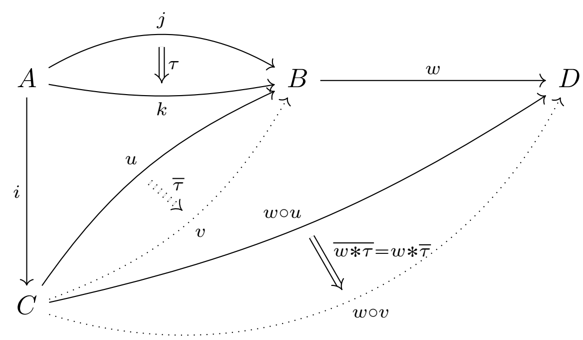


Definition 18 (Opextension)**.** Let $i : A \longrightarrow C$, and let $\tau : j \Rightarrow k$ be a natural transformation for functors $j, k : A \longrightarrow B$. Let $u : C \longrightarrow B$ be *under* j, in the sense that $u \circ i = j$. An *extension* of τ along i consists of a functor $v : C \longrightarrow B$ under k, and a $\overline{\tau} : u \Rightarrow v$ under τ, in the sense that $\overline{\tau} * i = \tau$.

Definition 19 (Basic opcofibration)**.** A *basic opcofibration* is a functor $i : A \longrightarrow C$ along which opextensions of any $\tau : j \Rightarrow k$ and $u : C \longrightarrow B$ under j exist. Moreover, these extensions are required to be

(1) *natural*, in that for $w : B \to D$ we have $\overline{w * \tau} = w * \overline{\tau}$, and
(2) *normalised*, in that $\overline{1_j} = 1_u$.

In pictures:

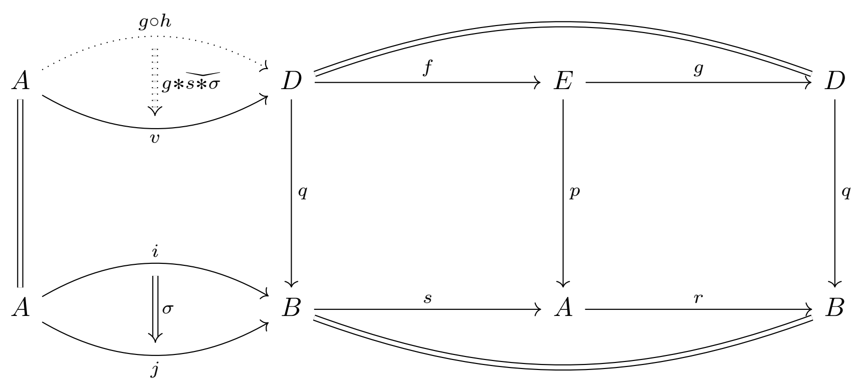


1.3.4. *Closure under retraction.* We may now obtain the result that we need to show that basic (op/co)fibrations can form classes of morphisms in WFSs on **Cat**.

Lemma 7.

(1) *The class of basic fibrations is closed under retraction.*
(2) *The class of basic cofibrations is closed under retraction.*
(3) *The class of basic opfibrations is closed under retraction.*
(4) *The class of basic opcofibrations is closed under retraction.*

Proof. We prove the first, for the others are similar. Suppose



Given v over j and $\sigma : i \Rightarrow j$, we whisker with s to obtain

$$s * \sigma : s \circ i \Rightarrow s \circ j$$

Then $p \circ (f \circ v) = s \circ q \circ v = s \circ j$, so we can lift $s * \sigma$ to

$$\widetilde{s * \sigma} : h \Rightarrow f \circ v$$

for some $h : A \to E$ over $s \circ i$, with $p * \widetilde{s * \sigma} = s * \sigma$. Then

$$g * \widetilde{s * \sigma} : g \circ h \Rightarrow g \circ f \circ v$$

But $g \circ f \circ v = v$, which is over j (w.r.t q), and

$$q \circ (g \circ h) = r \circ p \circ h = r \circ s \circ i = i$$

so $g \circ h$ is over i (w.r.t q), and

$$q * (g * \widetilde{s * \sigma}) = (q \circ g) * \widetilde{s * \sigma} = (r \circ p) * \widetilde{s * \sigma} = r * (p * \widetilde{s * \sigma}) = r * (s * \sigma) = \sigma$$

It is clear that this construction is natural. □

Proposition 12. *If $p : E \to B$ has the* enriched right lifting property (ERLP) *with respect to all future sections, then p is a basic fibration.*

Proof. This is explained in §1.3.1; in particular, having the ERLP with respect the injector $i_1 : \mathbb{1} \longrightarrow \mathbb{2}$ of the strict reflection that includes the trivial category into the codomain of the unique arrow of $\mathbb{2}$ is exactly the definition of a basic fibration (we are free to pick that the identity lifts to the identity). □

1.4. **The third and fourth factorisation systems.** We seek to factor any functor $f : A \to B$ as

$$\begin{array}{ccc} A & \xrightarrow{\quad j \quad} & E^f \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle r} & \\ & B & \end{array}$$

where j is a cofibration, and r is a future retraction.

The construction of E^f when A and B are groupoids is particularly simple, and is given also in Joyal and Tierney (2008, §2.2). The more general case of arbitrary categories has been described in a blog post of M. A. Shulman (2012).

The objects of the category E^f are the disjoint union of the objects of A and B. The hom-sets are defined by

$$E^f(x, y) \overset{\text{def}}{=} \begin{cases} A(x, y) & \text{if } x, y \in A \\ B(fx, y) & \text{if } x \in A, y \in B \\ B(x, y) & \text{if } x, y \in B \end{cases}$$

The category E^f has a very specific shape. It contains both A and B as full and faithful subcategories. These two subcategories are 'connected' by paths drawn from B: any $p : f(a) \to b \in B$ is a path from $a \in A$ to $b \in B$ in E^f. Composition is easy to define: within the subcategories A and B, it is directly inherited from them; composing p with $v : b \to b'$ is simply composition in B; and composing p with $u : a' \to a \in A$ is

$$p \circ_{E^f} u \overset{\text{def}}{=} p \circ_B f(u) : E(a', b)$$

The functor $j : A \hookrightarrow E^f$ includes the category A into the 'A part' of E^f. In §2.4 we will prove—amongst other things—it is a basic cofibration. Intuitively, this is easy to see: given any natural transformation $\tau : j \Rightarrow k$ with $v : E^f \longrightarrow C$ under k, we may define $u : E^f \longrightarrow C$ to be precisely j on the 'A part' of E^f, and v everywhere else. Then there is a natural transformation $\overline{\tau} : u \Rightarrow v$, which follows τ on the 'A part,' and is the identity on v everywhere else.

We now ought to define the reflector $r : E^f \longrightarrow B$: on the '$A$ part' it should act as f, and it should leave things put everywhere else. It follows that $r \circ j = f$.

There is a functor $i : B \hookrightarrow E^f$ that includes B into the 'B part,' and clearly $r \circ i = \mathsf{Id}_B$. We can then construct a natural transformation

$$\eta : \mathsf{Id} \Rightarrow i \circ r$$

For $x \in A$, $\eta_x : x \to f(x)$ 'crosses' from $x \in A$ to $f(x) \in B$. Hence, it must be an arrow in $B(f(x), f(x))$, and we can pick id_{fx}. For $y \in B$, $\eta_y \overset{\text{def}}{=} id_y : y \to y$.

It is not hard to check the triangle identities, and to hence verify that

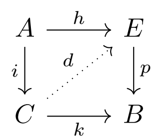
Proposition 13. *$r \dashv i$ is a strict reflection.*

Therefore, by Lemma 2, r is a future retraction.

Of course, as noted by M. A. Shulman (2012), a dual construction yields a category E_f, along with an inclusion $j' : A \hookrightarrow E_f$ that is an opcofibration. There is a $r' : E_f \longrightarrow B$ such that $r' \circ j' = f$. In addition, there is a $i' : B \hookrightarrow E_f$ with $i' \dashv r'$ being a strict coreflection. By Lemma 2 it follows that r' is a past retraction.

1.5. **Fillers.**

Theorem 2. *If i is a future section, p is a basic fibration, and*

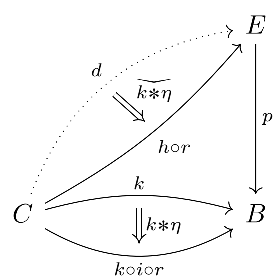


commutes, there is a diagonal filler d that makes the diagram commute.

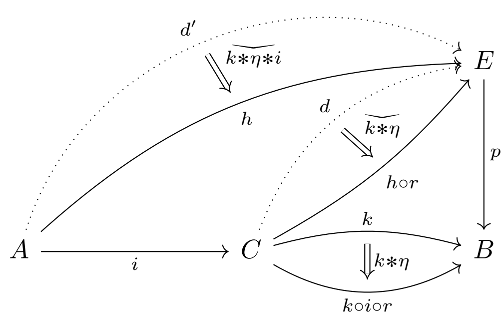
Proof. Let $r \circ i = \mathsf{Id}$ and $\eta : \mathsf{Id}_C \Rightarrow i \circ r$, so $k * \eta : k \Rightarrow k \circ i \circ r$. But as

$$p \circ (h \circ r) = k \circ i \circ r$$

we can lift $k * \eta$ along the fibration:



This gives us the requisite filler d, with $p \circ d = k$. It remains to show that $d \circ i = h$; but—noting that $h \circ r \circ i = h$—we can also whisker $k * \eta$ with i to get



As p is a basic fibration, we have the following naturality equation:

$$\widetilde{k * \eta} * i = \widetilde{k * \eta * i} : d' \Rightarrow h$$

Thus $d \circ i = d'$. But recall that, by the definition of future section, $\eta * i = 1_i$, so

$$\widetilde{k * \eta} * i = \widetilde{k * \eta * i} = \widetilde{k * 1_i} = \widetilde{1_{k \circ i}} = 1_h : d' \Rightarrow h$$

which is an identity, hence $d \circ i = d' = h$. □

Theorem 3. *If i is a past section, p is a basic opfibration, and*

$$\begin{array}{ccc} A & \xrightarrow{h} & E \\ {\scriptstyle i}\big\downarrow & \overset{d}{\nearrow} & \big\downarrow{\scriptstyle p} \\ C & \xrightarrow[k]{} & B \end{array}$$

commutes, there is a diagonal filler d that makes the diagram commute.

Proof. Entirely dual (op) to Theorem 2. □

Along with the results of §§1.1–1.3, these prove that

Theorem 4.

(1) $(\mathcal{FS}, \mathcal{F})$ *is a weak factorisation system on* **Cat**.
(2) $(\mathcal{PS}, \mathcal{F}^{op})$ *is a weak factorisation system on* **Cat**.

Theorem 5. *If j is a basic cofibration, and r is a future retraction, and*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle j}\big\downarrow & \overset{d}{\nearrow} & \big\downarrow{\scriptstyle r} \\ C & \xrightarrow[k]{} & D \end{array}$$

commutes, there is a diagonal filler d that makes the diagram commute.

Proof. Entirely dual (co) to Theorem 2. □

Theorem 6. *If j is a basic opcofibration, and r is a past retraction, and*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle j}\big\downarrow & \overset{d}{\nearrow} & \big\downarrow{\scriptstyle r} \\ C & \xrightarrow[k]{} & D \end{array}$$

commutes, there is a diagonal filler d that makes the diagram commute.

Proof. Entirely dual (op+co) to Theorem 2. □

Along with the results of §§1.2–1.3, 1.4 these prove that

Theorem 7.

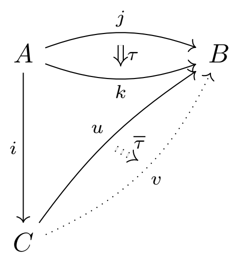
(1) $(\mathcal{C}, \mathcal{FR})$ *is a weak factorisation system on* **Cat**.
(2) $(\mathcal{C}^{op}, \mathcal{PR})$ *is a weak factorisation system on* **Cat**.

1.6. **Addendum: uncanny inclusions.** In this section we show that there are certain nontrivial inclusions between the classes involved in the North structure on **Cat**. To begin,

Proposition 14.

(1) *Every future section is a basic opcofibration.*
(2) *Every past section is a basic cofibration.*

Proof. We prove (1), with (2) being analogous. Suppose $r \circ i = \mathsf{Id}$ and $\eta : \mathsf{Id} \Rightarrow i \circ r$. Then, given the diagram



We define

$$v \stackrel{\text{def}}{=} C \xrightarrow{r} A \xrightarrow{k} B$$

so that $v \circ i = k$. We have that

$$u * \eta : u \Rightarrow u \circ i \circ r = j \circ r$$
$$\tau * r : j \circ r \Rightarrow k \circ r = v$$

and so we let

$$\overline{\tau} \stackrel{\text{def}}{=} (\tau * r) \circ (u * \eta) : u \Rightarrow v$$

Then

$$\overline{\tau} * i = (\tau * (r \circ i)) \circ (u * (\eta * i)) = \tau$$

as $r \circ i = \mathsf{Id}$, and $\eta * i = 1$, which is because i is a future section. ☐

On the fibrational side, we have

Proposition 15.

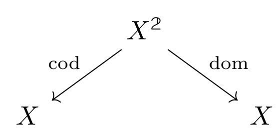
(1) *Every future retraction is a basic opfibration.*
(2) *Every past retraction is a basic fibration.*

The proofs are symmetric to Prop. 14.

2. Two-sided structure

The North structure on **Cat** provides a fair amount of machinery that may be used in the same way as that of model categories, viz. to providing diagonal fillers, notions of fibrance and cofibrance, etc. However, it stops slightly short of being able to reproduce standard arguments that are used in the construction of a meiotopy category. For example, there is no ostensible way to prove that, say, left meiotopy implies right meiotopy.

The main obstruction to such arguments is the one that we identified in §0.5.2, i.e. that the span constructed from the category of arrows $X^{\mathbb{2}}$ and its endpoint functors, namely



is not a fibration itself, and hence one cannot lift with respect to it.

However, it happens to be a *two-sided discrete fibration.* Such artefacts are spans whose legs are respectively an opfibration and a fibration that 'play well together.' The notion of 'well' used here has many equivalent formulations. Its main characteristics are that the *doubly-indexed fibres*—which contain arrows vertical with respect to both fibration and opfibration—are *discrete*. Moreover, each arrow in the total category can be factorised as two lifts. In that sense, two-sided discrete fibrations feel like two dimensional coordinate systems. We introduce some facts about them in §2.1, and provide factorisations in which they participate in §2.2.

It so happens that two-sided discrete fibrations correspond to another categorical contraption, namely that of *profunctor*. Profunctors are one of many categorifications of the notion of relation. We briefly discuss this not-so-well-known connection in §2.3. This correspondence has a dual aspect, in which profunctors are seen to correspond to *two-sided codiscrete cofibrations*. We discuss these in §2.4, followed by relevant factorisations in §2.5.

We then look at the types of diagonal fillers that lift against both two-sided (co)discrete (co)fibrations. These happen to be certain spans and cospans, which we call *cyclical sections* and *cyclical retractions* (§2.6). These lead us to slightly unusual diagonal fillers, which we investigate in §2.7.

2.1. **Two-sided discrete fibrations.** Two-sided discrete fibrations appear to have been introduced by Street (1980). Some very useful facts about them have been collected by Loregian and Riehl (2018). To begin:

Definition 20 (Two-sided discrete fibrations)**.** A span

$$\begin{array}{ccc} & E & \\ {\scriptstyle q}\swarrow & & \searrow{\scriptstyle p} \\ A & & B \end{array}$$

is a *two-sided discrete fibration* whenever

(1) any $u : q(e) \to a' \in A$ has a *unique* p-vertical q-lift, i.e. there is a unique $\widehat{u} : e \to e' \in E$ with domain e such that both $q(\widehat{u}) = u$ and $p(\widehat{u}) = id_{p(e)}$;
(2) any $v : b' \to p(e) \in B$ has a *unique* q-vertical p-lift, i.e. there is a unique $\breve{v} : e' \to e \in E$ with codomain e such that both $p(\breve{v}) = u$ and $q(\breve{v}) = id_{q(e)}$; and
(3) every morphism of E can be written as the composite of the two lifts: for each $f : e \to e' \in E$, cod $\widehat{q(f)}$ = dom $\breve{p(f)}$, and

$$e \xrightarrow{f} e' = e \xrightarrow{\widehat{q(f)}} \cdot \xrightarrow{\breve{p(f)}} e'$$

Conditions (1) and (2) say that p is 'fibration-like,' q is 'opfibration-like,' and moreover all lifts are 'cross-vertical' (fibration-like lifts are vertical with respect to q, and vice versa). Moreover, these lifts, along with the 'reindexed' objects e', are unique. In a minute we will see that these conditions suffice to ensure that q is a Grothendieck opfibration and p is a Grothendieck fibration.

Condition (3) is a little unusual, but it becomes a little more perspicuous through the following proposition.

Proposition 16. *In a two-sided fibration*

$$\begin{array}{ccc} & E & \\ {\scriptstyle q}\swarrow & & \searrow{\scriptstyle p} \\ A & & B \end{array}$$

there is at most one morphism of type $e \to e'$ above both $u : q(e) \to a \in A$ and $v : b \to p(e') \in B$.

Proof. Suppose $f : e \to e'$ with $q(f) = u$ and $p(f) = v$. Then

$$e \xrightarrow{f} e' = e \xrightarrow{\widehat{q(f)}} \cdot \xrightarrow{\widecheck{p(f)}} e' = e \xrightarrow{\hat{u}} \cdot \xrightarrow{\check{v}} e'$$

□

Given $a \in A$ and $b \in B$, define the *doubly-indexed fibre category* $E_{a,b}$ to be the subcategory of morphisms $f \in E$ such that $q(f) = id_a$ and $p(f) = id_b$. Evidently, Prop. 16 implies that $E_{a,b}$ is *discrete*: as $p(id) = id$ and $q(id) = id$, every morphism of $E_{a,b}$ must be an identity morphism.

We also record the following very useful fact.

Proposition 17. *Let $v : b' \to p(e) \in B$ lift q-vertically to $\check{v} : e' \to e$, and let $u : b'' \to p(e')$ lift q-vertically to $\check{u} : e'' \to e'$. Then*

$$\widecheck{v \circ u} = \check{v} \circ \check{u}$$

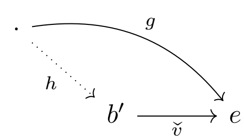
Proof. Both $\widecheck{v \circ u}$ and $\check{v} \circ \check{u}$ are above $v \circ u$ and q-vertical, and have codomain e. Thus, by the definition of two-sided fibration, $\mathrm{dom}(\widecheck{v \circ u}) = \mathrm{dom}(\check{v} \circ \check{u})$, and they are equal. □

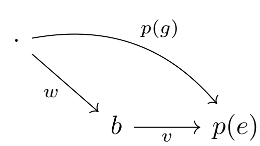
This leads us to the following alternative conception of two-sided discrete fibrations:

Theorem 8. $A \xleftarrow{q} E \xrightarrow{p} B$ *is a two-sided discrete fibration iff*

(1) *there exist p-vertical q-opcartesian lifts,*
(2) *there exist q-vertical p-cartesian lifts, and*
(3) *each doubly-indexed fibre $E_{a,b}$ is a discrete category.*

Proof. Let $A \xleftarrow{q} E \xrightarrow{p} B$ be a two-sided discrete fibration. We first show that for each $v : b \to p(e)$ the unique q-vertical p-lift $\check{v} : b' \to e$ is cartesian. Suppose



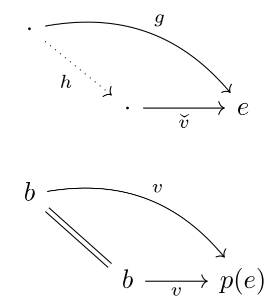


We need to show that there is a unique h above w that makes the top triangle commute. First, if such a h exists, it is unique: we have that $q(g) = q(\check{v} \circ h) = q(h)$, as $\check{v}$ is vertical with respect to q. Thus h is both above w and $q(g)$, and there is at most one such arrow by Prop. 16. But

$$g = \widecheck{p(g)} \circ \widehat{q(g)} = \widecheck{v \circ w} \circ \widehat{q(g)} = \check{v} \circ \check{w} \circ \widehat{q(g)}$$

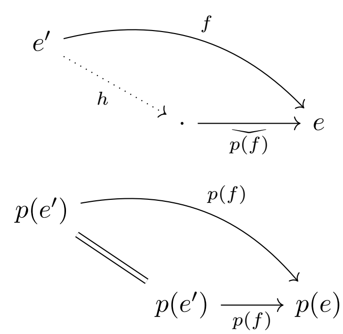
by Prop. 17. Thus, taking $h \stackrel{\text{def}}{=} \check{w} \circ \widehat{q(g)}$ completes the argument, as we have $p(h) = w$ and $\check{v} \circ h = g$. A similar proof applies to the p-vertical q-lifts, and we have already argued that $E_{a,b}$ is discrete.

Conversely, suppose (1)-(3) hold. We will prove that there is a unique q-vertical p-lift above $v : b \to p(e)$. Suppose



where $\check{v}$ is a q-vertical p-cartesian lift of v, and g is any q-vertical p-lift of v. There exists a unique p-vertical h that makes the top triangle commute. But if we apply q to the top triangle, we obtain $id = q(g) = q(\check{v}) \circ q(h) = q(h)$. Thus h is both p-vertical and q-vertical, so it can only be the identity; hence $g = \check{v}$, and their domains are equal. Similarly, we obtain unique p-vertical q-lifts $\hat{u}$ for each $u : q(e) \to a \in A$.

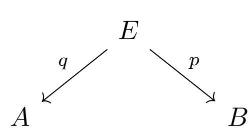
It remains to prove that we can factorise any morphism into its two lifts. Given $f : e \to e' \in E$, we have



We obtain a p-vertical h such that the top triangle commutes. But if we apply q to it, we obtain $q(f) = q(\widecheck{p(f)}) \circ q(h) = q(h) : q(e') \to q(e)$. So h is a p-vertical q-lift of $q(f)$, with domain e'. Hence, it must be equal to $\widehat{q(f)}$. □

As a result,

Corollary 2. *If*



is a two-sided discrete fibration, then q is a Grothendieck opfibration, and p is a Grothendieck fibration.

Remark 3. Bénabou (2000, §6.4) implicitly claims to characterise two-sided fibrations as those spans for which (1') q is a Grothendieck opfibration, (2') p is a Grothendieck fibration, and (3) $E_{a,b}$ is discrete. While by the above corollary we know that (1) and (2) do imply (1') and (2'), we cannot see how (1'-3') imply

(1) and (2): we certainly have (op)cartesian lifts, but it is not clear why they are 'cross-vertical.'

Example 1. The central example of a two-sided discrete fibration is the path space span $\begin{array}{ccc} & X^2 & \\ {}^{\mathrm{cod}}\swarrow & & \searrow^{\mathrm{dom}} \\ X & & X \end{array}$ for any category X. Let there be a morphism $\begin{array}{c} x \\ \downarrow{\scriptstyle\alpha} \\ y \end{array}$, considered as an *object* of X^2. Also, let there be a morphism $f : x' \to x \in X$, where it so happens that α is above the codomain with respect to the fibration dom, i.e. $\mathrm{dom}(\alpha) = x$. There is a unique morphism into α that is both above f w.r.t to dom, and also vertical w.r.t cod, namely

$$\begin{array}{ccc} x' & \xrightarrow{f} & x \\ {\scriptstyle\alpha\circ f}\,\vdots & & \downarrow{\scriptstyle\alpha} \\ y & = & y \end{array}$$

A similar situation is the case with dom and cod swapped: the unique morphism above $g : y \to y'$ w.r.t to cod and vertical with respect to dom is

$$\begin{array}{ccc} x & = & x \\ {\scriptstyle\alpha}\downarrow & & \vdots\,{\scriptstyle g\circ\alpha} \\ y & \xrightarrow[g]{} & y' \end{array}$$

It is now easy to see that we may factorise a morphism as

$$\begin{array}{ccc} x & \xrightarrow{f} & x' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ y & \xrightarrow[g]{} & y' \end{array} \quad = \quad \begin{array}{ccccc} x & = & x & \xrightarrow{f} & x' \\ {\scriptstyle\alpha}\downarrow & & \vdots & & \downarrow{\scriptstyle\alpha'} \\ y & \xrightarrow[g]{} & y' & = & y' \end{array}$$

by letting the dashed line be $\alpha' \circ f = g \circ \alpha$.

It is worth remarking that neither dom nor cod are discrete as (op)fibrations themselves: in fact,

$$\begin{array}{ccc} x' & \xrightarrow{f} & x \\ {\scriptstyle i^{-1}\circ\alpha\circ f}\,\vdots & & \downarrow{\scriptstyle\alpha} \\ y' & \xrightarrow[i]{\cong} & y \end{array}$$

is cartesian over f wrt cod for any isomorphism $i : y' \xrightarrow{\cong} y$.

2.1.1. *Lifting properties of two-sided discrete fibrations.* We would also like to characterise two-sided discrete fibrations as having *lifting properties* similar to those of basic (op)fibrations (§1.3), along with a particular condition, such as the lifting characterisation of Grothendieck fibrations given by Lemma 5.

Theorem 9. $\begin{array}{ccc} & E & \\ {}^{q}\swarrow & & \searrow^{p} \\ A & & B \end{array}$ *is a two-sided discrete fibration iff*

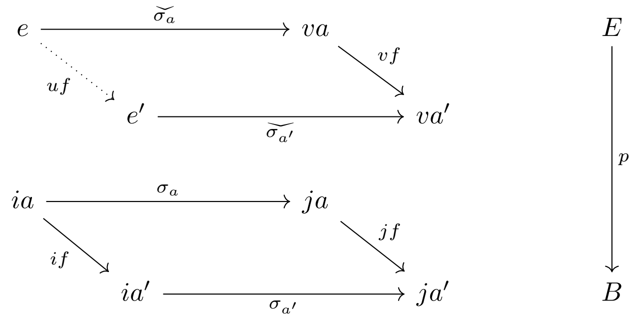
(1) *There is a unique q-vertical p-lift $\breve{\sigma} : u \Rightarrow v$ of $\sigma : i \Rightarrow j$, where $i, j : C \longrightarrow B$ and $v : C \longrightarrow E$ is over j w.r.t p.*

(2) *There is a unique p-vertical q-oplift $\widehat{\sigma} : u \Rightarrow v$ of $\sigma : i \Rightarrow j$, where $i, j : C \longrightarrow A$ and $u : C \longrightarrow E$ is over i w.r.t q.*
(3) *Every $\sigma : u \Rightarrow v$ for $u, v : C \longrightarrow E$ can be factorised as*

$$u \xRightarrow{\widehat{q*\sigma}} d \xRightarrow{\widetilde{p*\sigma}} v$$

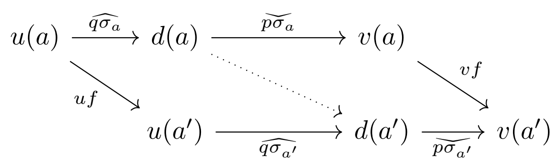
Proof. First we prove the backwards direction: taking $C = \mathbb{1}$, it is not hard to see that the above specialises to the usual definition of two-sided discrete fibration.

Now for the forwards direction. Consider the diagram

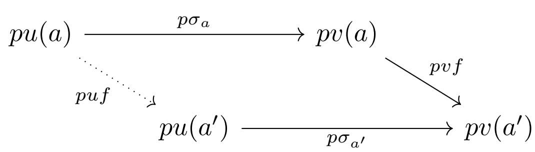


where $\breve{\sigma}_a$ is the unique q-vertical lift of σ_a, and so on. As we showed in Theorem 8, all such lifts are cartesian, so there exists a unique uf that makes the diagram commute, and uniqueness makes it functorial. (2) follows in a similar manner.

For (3), we factorise σ_a and $\sigma_{a'}$ (whilst ignoring the dotted arrow for a moment) as

(3) 

Focus on the right hand side open box, which w.r.t. p is over the open box



Taking puf as a lid for this box makes the diagram commute, as it is the p-image of a naturality square of σ. As $\widetilde{p\sigma_{a'}}$ is cartesian over $p\sigma_{a'}$, there is a unique arrow $h : d(a) \to d(a')$ that can go in the position of the dotted arrow in (3) that makes the right hand square commute. We record that $p(h) = pu(f)$, and $q(h) = qv(f)$; the latter we get by applying q to the right hand square of (3).

In an entirely symmetric way, we use the opcartesian arrow $\widehat{q\sigma_a}$ to obtain a $k : d(a) \to d(a')$ that makes the left hand square commute. But then $q(k) = qv(f)$ and $q(k) = pu(f)$, so in fact $k = h$. We pick this to be $d(f)$, and the uniqueness of this choice makes it functorial. Finally, placing it in the dotted position in (3) makes the entire diagram commute.

We thus obtain a factorisation of σ as $\widetilde{p\sigma} \circ \widehat{q\sigma}$. Before we conclude the proof, let us note for later use that $p \circ d = p \circ u$ and $q \circ d = q \circ v$. $\square$

2.1.2. *Commas and two-sided discrete fibrations.* Recall the definition of the comma category $\{f, g\}$ for functors $A \xrightarrow{f} X \xleftarrow{g} B$ (Definition 5 in §1.1.1). Its morphisms are commuting squares

$$\begin{array}{ccc} f(a) & \xrightarrow{f(\beta)} & f(a') \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha'} \\ g(b) & \xrightarrow[g(\gamma)]{} & g(b') \end{array} \tag{4}$$

In the general case, we can still define a functor $q : \{f, g\} \longrightarrow B$ by

$$\begin{array}{ccc} fx & \xrightarrow{f\beta} & fx' \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha'} \\ gy & \xrightarrow[g\gamma]{} & gy' \end{array} \qquad \overset{q}{\mapsto} \qquad x \xrightarrow{\beta} x'$$

and a functor $p : \{f, g\} \longrightarrow A$ by

$$\begin{array}{ccc} fx & \xrightarrow{f\beta} & fx' \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha'} \\ gy & \xrightarrow[g\gamma]{} & gy' \end{array} \qquad \overset{p}{\mapsto} \qquad y \xrightarrow{\gamma} y'$$

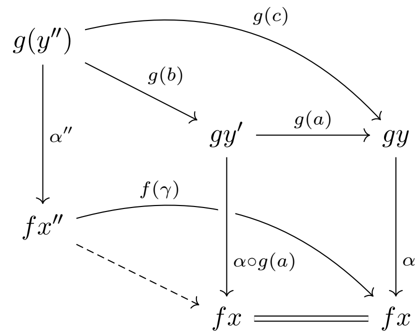
Indeed, these two functors specialize to the p and q of §1.1 in case either f or g is Id. What these functors do is provide us with a change of perspective: starting with diagram (4), we can take p and then f to obtain $fx \xrightarrow{f\beta} fx'$; or we can take q and then g to obtain $gy \xrightarrow{g\gamma} gy'$ again. The *objects* $\begin{array}{c} fx \\ \big\downarrow{\scriptstyle\alpha} \\ gy \end{array} \in \{f, g\}$ can then be seen as components of a natural transformation, indexed by themselves: they fill the diagram

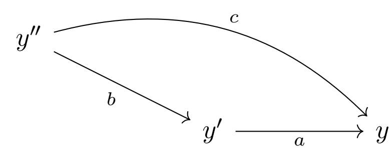
$$\begin{array}{ccc} \{f, g\} & \xrightarrow{p} & A \\ {\scriptstyle q}\big\downarrow & \Leftarrow & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & X \end{array}$$

We learned the following result from Loregian and Riehl (2018, Thm. 2.3.3), where it is attributed to Street (1974).

Theorem 10 (Street)**.** *The span* $\begin{array}{ccc} & \{f, g\} & \\ {\scriptstyle q}\swarrow & & \searrow{\scriptstyle p} \\ B & & A \end{array}$ *is a two-sided discrete fibration. Morever, every two-sided discrete fibration arises through such a comma category construction.*

We show the proof of the forward direction, which is simple. It is in fact a very mild generalization of Proposition 5 (see §1.1.1). To start, the following diagram suffices to convince us that (a, id_x) is cartesian over a:





The only object over y' that can make (a, id_x) an arrow from itself to α is the composite $\alpha \circ g(a)$. Once that is forcibly picked, there is again only one possibility for the horizontal dashed arrow, namely $f(\gamma)$. But then (a, id_x) is a q-vertical p-cartesian lift of a, and a similar story can be told about q. It is trivial to see that the doubly-indexed fibre $\{f, g\}_{a,b}$ is discrete, as commutation of the diagram

$$\begin{array}{ccc} f(a) & = & f(a') \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ g(b) & = & g(b') \end{array}$$

forces $\alpha = \alpha'$. Hence, by Theorem 8, the span induced by the comma category is a two-sided discrete fibration.

2.2. **Graphs, opgraphs, and factorisations.** There is, of course, a factorisation that involves two-sided discrete fibrations. This time it does not apply to functors themselves, but to their *(op)graphs*.

Definition 21 (Graph and opgraph of a functor)**.** Let $f : A \longrightarrow B$ be a functor.

(1) The *graph of* f is the functor $\langle \mathsf{Id}_A, f \rangle : A \longrightarrow A \times B$.
(2) The *opgraph of* f is the functor $\langle f, \mathsf{Id}_A \rangle : A \longrightarrow B \times A$.

Given $f : A \longrightarrow B$, recall the functor $i : A \longrightarrow \{B, f\}$ from §1.1.2, which is defined by sending $\gamma : x \to x' \in A$ to

$$\begin{array}{ccc} fx & \xrightarrow{f\gamma} & fx' \\ \| & & \| \\ fx & \xrightarrow[f\gamma]{} & fx' \end{array}$$

Recall also that the action of the fibration $p : \{B, f\} \longrightarrow A$ on the above square is to return $\gamma : x \to x'$. Moreover, the action of the opfibration $q : \{B, f\} \longrightarrow B$ is to return $f\gamma : fx \to fx'$. That is, we have a factorisation

$$\begin{array}{ccc} A & \xrightarrow{\quad i \quad} & \{B, f\} \\ & {\scriptstyle\langle f, \mathsf{Id}_A\rangle} \searrow \quad \swarrow {\scriptstyle\langle q, p\rangle} & \\ & A \times B & \end{array}$$

But recall that, by Prop. 6, $q \dashv i$ is a strict reflection. Thus, we have factorised the opgraph of f into the right adjoint of a strict reflection (which, by Lemma 2, is also a future section) followed by a two-sided discrete fibration. Similarly, we have

$$\begin{array}{ccc} A & \xrightarrow{\quad j \quad} & \{f, B\} \\ & {\scriptstyle\langle \mathsf{Id}_A, f\rangle} \searrow \quad \swarrow {\scriptstyle\langle q', p'\rangle} & \\ & B \times A & \end{array}$$

So, the graph of f can be factorised into the left adjoint of a strict coreflection (hence also a past section) followed by a two-sided discrete fibration.

The most interesting case of the above construction is the (op)graph of $\mathsf{Id}_A : A \longrightarrow A$, which is of course the diagonal $\Delta_A \stackrel{\text{def}}{=} \langle \mathsf{Id}_A, \mathsf{Id}_A \rangle : A \to A \times A$. It is actually a very familiar one:

$$\begin{array}{ccc} A & \xrightarrow{\quad \mathsf{refl}_A \quad} & A^2 \\ & {\scriptstyle\Delta_A} \searrow \quad \swarrow {\scriptstyle\langle \mathrm{cod}, \mathrm{dom}\rangle} & \\ & A \times A & \end{array}$$

The functor $\mathsf{refl}_A : A \longrightarrow A^2$ is as usual defined by sending $\gamma : x \to x' \in A$ to

$$\begin{array}{ccc} x & \xrightarrow{\gamma} & x' \\ \| & & \| \\ x & \xrightarrow[\gamma]{} & x' \end{array}$$

It follows that refl_A is both a future section and a past section. In fact, the components of this factorisation are particularly interesting, as

$$\mathrm{cod} \dashv \mathsf{refl}_A \dashv \mathrm{dom}$$

where the first adjunction is a strict reflection, and the second is a strict coreflection. This implies that refl_A is a *cyclical section*: see §2.6, and Example 3.

2.3. **Interlude: profunctors.** It so happens that there is an exact correspondence between the concept we just introduced, namely two-sided discrete fibrations, and the well-known concept of a *profunctor*. In fact, this correspondence is threefold, in that

- two-sided discrete fibrations
- profunctors
- two-sided codiscrete cofibrations

are equivalent presentations of the same kind of contraption. First things first:

Definition 22 (Profunctor)**.** A *profunctor* $\phi : A \nrightarrow B$ from A to B is a functor

$$\phi : A^{\mathrm{op}} \times B \longrightarrow \mathbf{Set}$$

Suppose for a moment that A and B are sets (and hence that $A^{\mathrm{op}} = A$). Then ϕ assigns to each pair of objects $a \in A$, $b \in B$ a set $\phi(a, b)$. This set can be construed as the set of *evidence* that a is related to b by ϕ. If it is empty, then a is not related to b. In any other case it is, but there may be many non-trivial *witnesses* to that.

In the more general case of A and B being categories, then ϕ also provides an *action* of morphisms on relation evidence. For example, if $e \in \phi(a, b)$ and $f : a' \to a \in A$, then

$$e \cdot f \stackrel{\text{def}}{=} \phi(f, id_b)(e) \in \phi(a', b)$$

is a witness that a' and b are related. Similarly, if $g : b \to b'$ we have that

$$g \cdot e \stackrel{\text{def}}{=} \phi(id_a, g)(e) \in \phi(a, b')$$

Functoriality, which in the above notation takes the simple form

$$e \cdot (f_2 \circ f_1) = (e \cdot f_2) \cdot f_1$$
$$(g_2 \circ g_1) \cdot e = g_2 \cdot (g_1 \cdot e)$$
$$(g \cdot e) \cdot f = g \cdot (e \cdot f)$$

serves to ensure that this action is coherent with respect to A and B. Profunctors form both a category, but also the paradigmatic example of a *bicategory*. We only mention one interesting fact, viz. that the identity profunctor $\mathsf{id}_X : X \nrightarrow X$, which stands for the identity relation on a category X, is the hom functor

$$X(-, -) : X^{\mathrm{op}} \times X \longrightarrow \mathbf{Set}$$

References include Borceux (1994, §7.7), Bénabou (2000), and Joyal's CatLab.[3]

2.3.1. *The co-collage.* One can construct a two-sided discrete fibration from a profunctor, which is a kind of Grothendieck construction. Given $\phi : A \nrightarrow B$, we let $\int \phi$ be the category with

- **objects:** $(a \in A, b \in B, e \in \phi(a, b))$
- **morphisms:** $(f, g) : (a, b, e) \to (a', b', e')$ are pairs of morphisms $f : a \to a'$ and $g : b \to b'$ such that $g \cdot e = e' \cdot f$.

That is: (f, g) is a morphism from $e \in \phi(a, b)$ to $e' \in \phi(a', b')$ just if pulling e' back along $f : a \to a'$ to get $e' \cdot f \in \phi(a, b')$ yields the same result as pushing e forwards along $g : b \to b'$ to obtain $g \cdot e \in \phi(a, b')$. Then, the span defined by

$$\begin{array}{ccc} & \int \phi & \\ {\scriptstyle q}\swarrow & & \searrow{\scriptstyle p} \\ B & & A \end{array} \qquad \begin{array}{ccc} & e \in \phi(a, b) & \\ {\scriptstyle q}\swarrow & & \searrow{\scriptstyle p} \\ b & & a \end{array}$$

is a two-sided discrete fibration. For example, pulling back along $f : a \to a'$ provides a lift

$$\begin{array}{ccc} e \cdot f \in \phi(a, b) & \xrightarrow{(f, id_b)} & e \in \phi(a', b) \\ & & \\ a & \xrightarrow{\quad f \quad} & a' \end{array} \qquad \begin{array}{c} \int \phi \\ \downarrow{\scriptstyle p} \\ A \end{array}$$

[3] At the time of writing (Nov 2018) many equations on the CatLab are not rendered correctly.

which is unique above f with respect to p, and vertical with respect to q. I often call this construction the *co-collage of* ϕ, but that is not established terminology.

There is also a way to construct a profunctor from a two-sided discrete fibration, by mapping (a, b) to the doubly-indexed fibre $E_{a,b}$ which—as we showed in Theorem 8—is discrete, hence a set. This strengthens the correspondence between these two contraptions to an equivalence: see (Loregian and Riehl, 2018, Theorem 2.3.2).

Remark 4. We note that the co-collage of the identity profunctor $\mathsf{id}_X : X \looparrowright X$ is actually the paradigmatic two-sided discrete fibration $\langle \text{cod}, \text{dom} \rangle : X^{\mathbb{2}} \longrightarrow X$. *Thus, we obtain the path space from the identity profunctor.*

2.3.2. *The collage.* Far more well-known than the aforementioned perspective is the construction of the *collage* of a profunctor $\phi : A \looparrowright B$. It is the category $A \star_\phi B$ whose objects are the disjoint union of objects of A and B, with morphisms are

$$(A \star_\phi B)(x, y) \stackrel{\text{def}}{=} \begin{cases} A(x, y) & \text{if } x, y \in A \\ \phi(x, y) & \text{if } x \in A,\ y \in B \\ B(x, y) & \text{if } x, y \in B \end{cases}$$

The collage has a useful pictorial representation, which looks very much like a kind of cylinder:

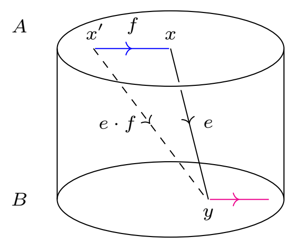


FIGURE 3. A collage

The 'top' of this structure is the category A, and the 'bottom' is the category B. The black diagonal arrow depicts an element $e \in \phi(a, b)$, which in this setting is sometimes called a *heteromorphism*. Near the top is a blue line, which stands for a morphism $f : x \to x' \in A$. The diagonal dashed arrow is the result of the contravariant action of f on e, namely $e \cdot f$. The magenta line at the bottom represents an arbitrary morphism of B.

In fact, Joyal shows that all such structures of the above shape correspond to profunctors. Because of that shape, they are called

Definition 23 (Barrels)**.** A *barrel* is a category X along with a functor

$$f : X \longrightarrow \mathbb{2}$$

The idea is that the fibre $f^{-1}(0)$ over 0, the *top* of the barrel,[4] stands for A in the above picture, whereas the fibre $f^{-1}(1)$ over q—the *bottom* of the barrel—stands for B. Moreover, the top of the barrel is a *sieve*, in that precomposing any arrow to an arrow at the top yields again an arrow at the top. The bottom of the barrel has the dual property, i.e. is a *cosieve*. The heteromorphisms are all then sent to

[4]What we call top Joyal calls bottom and vice versa.

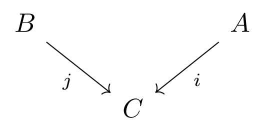
the walking arrow $0 \to 1 \in 2$. Joyal proves an equivalence between barrels and the category[5] of profunctors: up to equivalence, every barrel is a collage of some profunctor.

We note that it is not hard to show that the collage of the identity profunctor $\mathsf{id}_X : X \to\!\!\!\!\!\mapsto X$ is equivalent to $X \times 2$. *Thus, we obtain the cylinder of a space from the identity profunctor.*

2.3.3. *Two-sided codiscrete cofibrations.* Before seeing them as barrels, collages were characterised as *two-sided codiscrete cofibrations*, which are cospans, dual to what we had before. There is quite a bit of literature on these, beginning with (Street, 1980); see also Carboni et al. (1994) and Rosebrugh and Wood (1988) and the nLab entry on codiscrete cofibrations.. These are not so well known as structures, and we devote the entirety of the next section to them.

2.4. **Two-sided codiscrete cofibrations.** Two-sided codiscrete cofibrations are cospans of category, which are precisely dual (co) to two-sided discrete fibrations. The characterisation of the latter given in Theorem 9 immediately leads to the following definition.

Definition 24 (Two-sided codiscrete cofibration)**.** The cospan

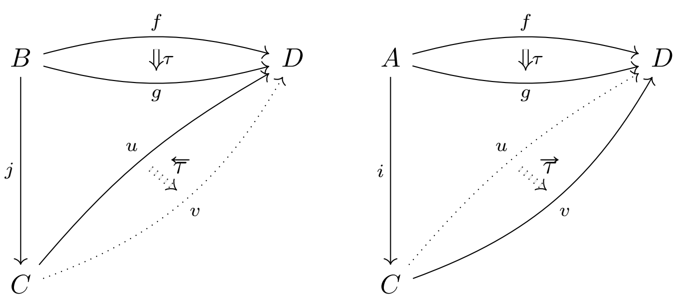


is a two-sided codiscrete cofibration just if

(1) For every $\tau : f \Rightarrow g$ where $f, g : A \longrightarrow D$ and any $v : C \longrightarrow D$ below g w.r.t. j there exists a *unique* j-covertical i-extension $\overrightarrow{\tau} : u \Rightarrow v$ of τ, i.e. a unique $\overrightarrow{\tau}$ such that both $\overrightarrow{\tau} * i = \tau$ and $\overrightarrow{\tau} * j = \mathsf{Id}_{v \circ j}$.
(2) For every $\tau : f \Rightarrow g$ where $f, g : A \longrightarrow D$ and any $u : C \longrightarrow D$ below f w.r.t. i there exists a *unique* j-covertical i-opextension $\overleftarrow{\tau} : u \Rightarrow v$ of τ, i.e. a unique $\overleftarrow{\tau}$ such that both $\overleftarrow{\tau} * j = \tau$ and $\overleftarrow{\tau} * i = \mathsf{Id}_{v \circ i}$.
(3) Every $\tau : u \Rightarrow v$ for $u, v : C \longrightarrow D$ can be factorised as

$$u \xRightarrow{\overleftarrow{\tau * j}} w \xRightarrow{\overrightarrow{\tau * i}} v$$

The diagrams are the usual pictures for basic cofibrations and opcofibrations:



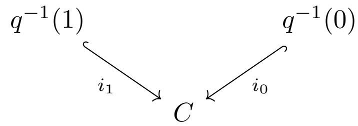
taking care to not forget the ‘cross-coverticality’ of each these (op)extensions.

We want to argue that two-sided codiscrete cofibrations and profunctors—both in **Cat**—correspond. This fact follows in the more general setting of profunctors valued in any symmetric monoidal category $\mathcal{V}$, as shown by Street (1980), and

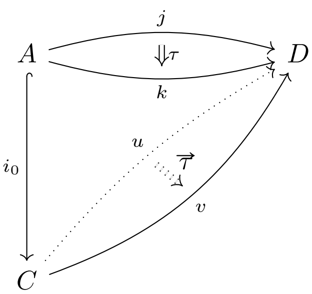
[5]Not to be confused with the bicategory of profunctors.

quoted by Loregian and Riehl (2018, §4.3). That is: the two-sided codiscrete fibrations in $\mathcal{V}$-**Cat** are exactly the collages of $\mathcal{V}$-profunctors. However, we want to present a more elementary version of this result. In one direction, we show that

Proposition 18. *Any barrel $q : C \longrightarrow 2$ induces a two-sided codiscrete cofibration, viz.*



Proof. Let $A \stackrel{\text{def}}{=} q^{-1}(0)$ and $B \stackrel{\text{def}}{=} q^{-1}(1)$. We show i_0 is a cofibration. Suppose



The barrel structure $q : C \longrightarrow 2$ on C partitions the morphisms into those in $A = q^{-1}(0)$, those in $B = q^{-1}(1)$, and the heteromorphisms that cross from the former to the latter. With that in mind, we define the necessary functor u by case analysis; we make it coincide with j on A, v on B, and use τ for heteromorphisms:

$$\begin{array}{lcl} u : C & \longrightarrow & D \\ \quad f : a \to a' \in A & \longmapsto & j(f) : j(a) \to j(a') \\ \quad g : b \to b' \in B & \longmapsto & v(f) : v(b) \to v(b') \\ \quad e : a \to b & \longmapsto & v(e) \circ \tau_a : j(a) \to v(b) \end{array}$$

Functoriality for the case of heteromorphisms follows from $v \circ i_0 = k$ and the naturality of τ in the third. One can extend τ to $\overleftarrow{\tau}$ by defining it to be τ on A, and identity elsewhere, i.e.

$$\overleftarrow{\tau}_a \stackrel{\text{def}}{=} \tau_a : j(a) \to k(a)$$
$$\overleftarrow{\tau}_b \stackrel{\text{def}}{=} id_b : v(b) \to v(b)$$

Evidently, this is covertical w.r.t. i_1. For a heteromorphism $e : a \to b$ we have that

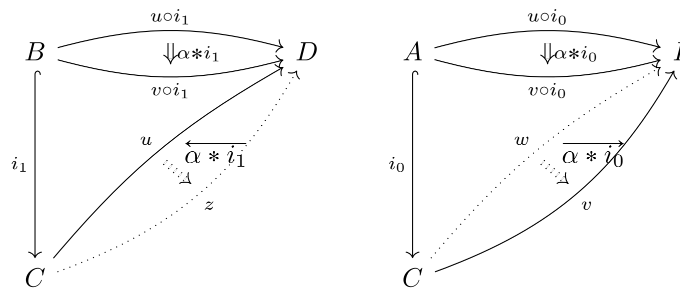
$$\begin{array}{ccc} j(a) & \xrightarrow{u(e)} & v(b) \\ {\scriptstyle \tau_a}\downarrow & & \Vert \\ k(a) & \xrightarrow[v(e)]{} & v(b) \end{array}$$

commutes by definition of $u(e) \stackrel{\text{def}}{=} v(e) \circ \tau_a$. Similarly, i_0 is an opcofibration.

It remains to show the factorisation property. Given $\alpha : u \Rightarrow v$ for $u, v : C \longrightarrow D$, we 'restrict' α to A and B by whiskering with i_0, i_1 to obtain

$$\alpha * i_0 : u \circ i_0 \Rightarrow v \circ i_0 : A \longrightarrow D$$
$$\alpha * i_1 : u \circ i_1 \Rightarrow v \circ i_1 : B \longrightarrow D$$

We can then extend those transformations to ones on C, as in the following diagram:



It is the case that $v = w$: chasing through the details of the extensions defined before, we have that both z and w coincide with u on A and with v on B. On heteromorphisms, w maps $e : a \to b$ to

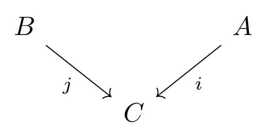
$$v(e) \circ (\alpha * i_0)_a = v(e) \circ \alpha_a$$

and z maps it to

$$(a * i_1)_b \circ u(e) = \alpha_b \circ u(e)$$

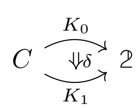
which are equal by the naturality of α. We can then obtain $\overrightarrow{\alpha * i_0} \circ \overleftarrow{\alpha * i_1}$, and see that it is componentwise equal to α. □

Proposition 19. *Given a two-sided codiscrete cofibration*



we can construct a barrel $q : C \longrightarrow 2$ *with* $j(B) \subseteq q^{-1}(0)$ *and* $i(A) \subseteq q^{-1}(1)$.

Proof. Let $K_0, K_1 : C \longrightarrow 2$ be the constant 0 and constant 1 functors respectively. It is easy to see that there is a natural transformation



with components $\delta_c = 0 \to 1$. We can factor this natural transformation into

$$\delta = K_0 \overset{\overleftarrow{\delta * j}}{\Longrightarrow} q \overset{\overrightarrow{\delta * i}}{\Longrightarrow} K_1$$

Hence, $q \circ i = K_0 \circ i = K_0$ and $q \circ j = K_1 \circ j = K_1$. Thus, every $i(f) : i(a) \to i(a')$ is in $q^{-1}(0)$ for $f : a \to a' \in A$, and $j(g) : j(b) \to j(b')$ is in $q^{-1}(1)$ for $g : b \to b'$. □

It is evident that if we begin with a barrel, use Proposition 18 to obtain a two-sided codiscrete cofibration, and then Proposition 19, we obtain the barrel we started with in the first place.

Remark 5. Nevertheless, the other direction is not at all evident: if we begin with an arbitrary two-sided codiscrete cofibration $B \xrightarrow{j} C \xleftarrow{i} A$, use Prop. 19 to get a barrel $q : C \longrightarrow 2$, and then take the cospan created by its fibres, as in Prop. 18, we obtain a cospan $q^{-1}(1) \xhookrightarrow{i_1} C \xhookleftarrow{i_0} q^{-1}(0)$ which is not evidently the same as the one with which we started. We still do have that $j(B) \subseteq q^{-1}(0)$ and $i(A) \subseteq q^{-1}(1)$, but this is as far as I can currently see.

2.5. **Cographs, opcographs, and factorisations.** As in §2.2, two-sided codiscrete fibrations participate in a factorisation. The arrows that will be factorised this time are duals to the ones before, namely

Definition 25 ((Op)cograph of a functor)**.** Let $f : A \longrightarrow B$ be a functor.

(1) The *cograph of* f is the functor $[f, \mathsf{Id}_B] : A + B \longrightarrow B$.
(2) The *opcograph of* f is the functor $[\mathsf{Id}_B, f] : B + A \longrightarrow B$.

Given a morphism $f : a \to b \in A$, one can immediately obtain a presheaf $A(-, f) : A^{\text{op}} \longrightarrow \mathbf{Set}$. In a similar manner, a functor $f : A \longrightarrow B$ gives rise to *two* profunctors. The first one is the profunctor

$$B(-, f(-)) : B^{\text{op}} \times A \longrightarrow \mathbf{Set}$$

which we denote by $\phi_f : B \twoheadrightarrow A$. The second one is the profunctor

$$B(f(-), -) : A^{\text{op}} \times B \longrightarrow \mathbf{Set}$$

which we denote by $\phi^f : A \twoheadrightarrow B$.

Using the results of §2.1 we see that, when seen as two-sided discrete fibrations, the profunctors ϕ_f and ϕ^f correspond the comma categories $\{B, f\}$ and $\{f, B\}$. Indeed, we notice that

$$ob\ \left(\int \phi^f\right) \stackrel{\text{def}}{=} (b \in B, a \in A, f \in B(f(a), b))$$

and similarly for $\int \phi_f$. Recall that this perspective led to the factorisation of arbitrary functors into future/past sections and two-sided discrete fibrations in §2.2, which were generalisations of the factorisations obtained in §1.1.

In an entirely dual manner, we may obtain a generalisation of the factorisations obtained in §1.4. The collage of the profunctor ϕ^f is

$$(A \star_{\phi^f} B)(x, y) \stackrel{\text{def}}{=} \begin{cases} A(x, y) & \text{if } x, y \in A \\ B(fx, y) & \text{if } x \in A,\ y \in B \\ B(x, y) & \text{if } x, y \in B \end{cases}$$

which is exactly the construction of E^f in §1.4. This collage can be equipped with the obvious barrel structure that sends A to 0, and B to 1. Hence, by using Prop.

18 we obtain a two-sided codiscrete cofibration

$$\begin{array}{ccccc} B & & & & A \\ & \overset{i}{\searrow} & & \overset{j}{\swarrow} & \\ & & A \star_{\phi f} B & & \end{array}$$

There is a functor $r : A \star_{\phi f} B \longrightarrow B$ defined by

$$\begin{aligned} u : a \to a' &\quad\longmapsto\quad f(u) : f(a) \to f(a') \\ e : f(a) \to b &\quad\longmapsto\quad e : f(a) \to b \\ v : b \to b' &\quad\longmapsto\quad v : b \to b' \end{aligned}$$

We then have that $r \circ i = \mathsf{Id}_B$, and $r \circ j = f$, and hence a factorisation

$$\begin{array}{ccc} A + B & \xrightarrow{[j,i]} & B \star_{\phi f} A \\ & {\scriptstyle [f,\mathsf{Id}_B]}\searrow \quad \swarrow{\scriptstyle r} & \\ & B & \end{array}$$

of the cograph, where the first factor is a two-sided codiscrete cofibration. The second factor is a future retraction; this we can deduce from the fact that

Proposition 20. *$r \dashv i$ is a strict reflection.*

Proof. $i : B \longrightarrow A \star_{\phi f} B$ is a full and faithful inclusion, so it remains to prove $r \dashv i$. We define $\eta : \mathsf{Id}_{A \star_{\phi f} B} \Rightarrow i \circ r$ by

$$\eta_a \overset{\text{def}}{=} id_{f(a)} : a \to f(a)$$
$$\eta_b \overset{\text{def}}{=} id_b : b \to b$$

The first of these equations is well-typed because an arrow $\eta_a : a \to f(a) \in A \star_{\phi f} B$ is really an arrow $f(a) \to f(a) \in B$, as $a \in A$ and $f(a) \in B$. It is simple to verify that this is natural, that $\eta * i = 1$ (on $b \in B$ it is identity), and that $r * \eta = 1$. □

2.6. **Cyclical sections and retractions.** In §1 we explored the North structure on **Cat**. This structure induced quite a few solutions to various lifting problems, e.g. that future sections have the left lifting property with respect to basic fibrations.

Our development in §2.1–2.4 begs the question of whether there are contraptions that have some kind of lifting property with respect to two-sided (co)discrete (co)fibrations. Surprisingly, the answer if positive. Nevertheless, the kind of lifting problem that we can solve is rather unusual, as it is *two-sided*.

Two-sided discrete fibrations are a kind of *span*. It transpires that the kind of object against which a span lifts is, in fact, a *cospan*. These cospans we will call *cyclical sections*. Symmetrically, two-sided codiscrete cofibrations are *cospans*, and the contraptions against which they lift are *spans*: they are called *cyclical retractions*.

In this section we shall introduce both of these kinds of objects, which are closely related to the future/past section/retraction pattern of §1. We will discuss their lifting properties in §2.7.

2.6.1. *Cyclical sections.* We define what it means for a cospan in **Cat** to be a cyclical section.

Definition 26 (Cyclical sections)**.** A cospan $D \xrightarrow{i_0} C \xleftarrow{i_1} D$ is a *cyclical section* whenever there exists a $r : C \longrightarrow D$ such that i_0 and i_1 are respectively a past section and a future section with respect to r.

Explicitly, $r \circ i_0 = r \circ i_1 = \mathsf{Id}_D$, and there exist

$$\eta : \mathsf{Id}_D \Rightarrow i_1 \circ r$$
$$\epsilon : i_0 \circ r \Rightarrow \mathsf{Id}_D$$

such that $\eta * i_0 = 1_i$ and $\epsilon * i_1 = 1_i$.

The adjective 'cyclical' is supposed to invoke the intuition that D can be included in C in a full and faithful manner at two different locations, one 'at the start' of C (i.e. in a coreflective manner), and one 'at the end' of C (i.e. in a reflective manner).

Indeed, as with future and past sections, many useful examples of cyclical sections come from adjoint cylinders

$$i_0 \dashv r \dashv i_1$$

where $i_0 \dashv r$ is a strict coreflection, and $r \dashv i_1$ is a strict reflection. In this case, Lemma 2 applies to show that i_0 is a past section (w.r.t. r) and i_1 is a future section (w.r.t. r again). This situation is commonly known as an adjoint triple.

Example 2. The main motivating example of a cyclical section is the *directed cylinder* $X \times \mathbb{2}$ of a category X. The cospan consists of the two inclusions

$$\begin{aligned} i_0 : X &\longrightarrow X \times \mathbb{2} \\ \gamma : x \to y &\longmapsto (\gamma, id_0) : (x, 0) \to (y, 0) \\ i_1 : X &\longrightarrow X \times \mathbb{2} \\ \gamma : x \to y &\longmapsto (\gamma, id_1) : (x, 1) \to (y, 1) \end{aligned}$$

In the opposite direction, we have the functor $r : X \times \mathbb{2} \longrightarrow X$, defined by

$$(\gamma, d) : (x, i) \to (y, j) \quad \longmapsto \quad \gamma : x \to y$$

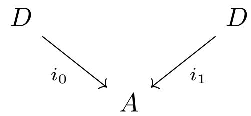
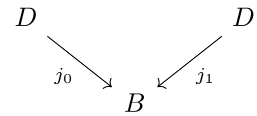
which 'collapses' the cylinder by forgetting the cylindrical dimension. We have

Proposition 21.

(1) $i_0 \dashv r$ *is a strict coreflection.*
(2) $r \dashv i_1$ *is a strict reflection.*

Hence, by Lemma 2, this cospan is a cyclical section.

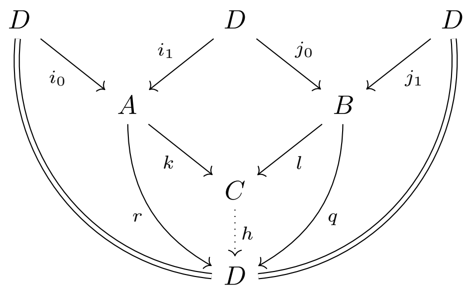
Proposition 22. *Cyclical sections are closed under cospan composition.*

Proof. Let there be cyclical sections  and ,

witnessed by retractions $r : A \longrightarrow D$, $q : B \longrightarrow D$, and natural transformations

$$\begin{aligned} \eta : \mathsf{Id}_A &\Rightarrow i_1 \circ r \\ \epsilon : i_0 \circ r &\Rightarrow \mathsf{Id}_A \\ \theta : \mathsf{Id}_B &\Rightarrow j_1 \circ q \\ \zeta : j_0 \circ q &\Rightarrow \mathsf{Id}_B \end{aligned}$$

We have that **Cat** is strictly 2-cocomplete as a 2-category, so we construct the composite by taking the strict 2-pushout $C \stackrel{\text{def}}{=} A +_D B$:



By the universal property of the pushout, we obtain a unique $h : C \longrightarrow D$ such that $h \circ (k \circ i_0) = h \circ (l \circ j_1) = \mathsf{Id}_D$. It remains to present natural transformations witnessing $[k \circ i_0, l \circ j_1] : D + D \longrightarrow C$ as a cyclical section (w.r.t. h). By the universal property of the strict 2-pushout, whenever given two natural transformations

$$\lambda_A : k \circ i_0 \circ h \circ k \Rightarrow k \quad \text{and} \quad \lambda_B : k \circ i_0 \circ h \circ l \Rightarrow l$$

such that

$$\lambda_A * i_1 = \lambda_B * j_0$$

we have a unique natural transformation

$$\lambda : (k \circ i_0) \circ h \Rightarrow \mathsf{Id}_C$$

such that $\lambda * k = \lambda_A$, and $\lambda * l = \lambda_B$. We have $k \circ i_0 \circ h \circ k = k \circ i_0 \circ k$, so let

$$\lambda_A \stackrel{\text{def}}{=} k * \epsilon : k \circ i_0 \circ r \Rightarrow k$$

Also, noticing $k \circ i_0 \circ h \circ l = k \circ i_0 \circ q = k \circ i_0 \circ r \circ i_1 \circ q$, we let

$$\lambda_B \stackrel{\text{def}}{=} k \circ i_0 \circ r \circ i_1 \circ q \xRightarrow{k * \epsilon * i_1 * q} k \circ i_1 \circ q = l \circ j_0 \circ q \xRightarrow{l * \zeta} l$$

It is easy to see that $\lambda_A * i_1 = \lambda_B * j_0 = k * \epsilon * i_0$. It remains to check that

$$\lambda * (k \circ i_0) = \lambda_A * i_0 = k * \epsilon * i_0 = 1$$

by the universal property, and because $\epsilon * i_0 = 1$. Rinse and repeat for the second arm of the cospan. □

2.6.2. *Cyclical retractions.* Symmetrically, certain spans in **Cat** are

Definition 27 (Cyclical retractions)**.** A span

$$\begin{array}{ccc} & X & \\ {\scriptstyle r_0}\swarrow & & \searrow{\scriptstyle r_1} \\ A & & A \end{array}$$

is a *cyclical retraction* whenever there exists a $i : A \longrightarrow X$ such that r_0 and r_1 are respectively a past retraction and a future retraction with respect to i.

Explicitly, $r_0 \circ i = r_1 \circ i_1 = \mathsf{Id}_A$, and there exist

$$\begin{aligned} \eta :\ \mathsf{Id}_X &\Rightarrow i \circ r_1 \\ \epsilon :\ i \circ r_0 &\Rightarrow \mathsf{Id}_X \end{aligned}$$

such that $r_0 * \epsilon = 1_{r_0}$ and $r_1 * \epsilon = 1_{r_1}$.

Example 3*.* Recall the path space span

$$\begin{array}{ccc} & A^{\mathbb{2}} & \\ {\scriptstyle \mathrm{dom}}\swarrow & & \searrow{\scriptstyle \mathrm{cod}} \\ A & & A \end{array}$$

as well as the functor $\mathsf{refl}_A : A \longrightarrow A^{\mathbb{2}}$. It is easy to show that

Proposition 23*.*

(1) *cod* $\dashv$ refl_A *is a strict reflection.*
(2) refl_A $\dashv$ *dom is a strict coreflection.*

These are generalizations of Proposition 6. By Lemma 2, they also imply that dom is a past retraction, and cod is a future retraction, both with respect to refl_A, and refl_A itself is both a past and future section.

2.7. **Two-sided fillers.** In §2.6 we introduced cyclical sections and cyclical retractions, which are certain classes of (co)spans. Here we shall show that these classes of (co)spans have a 'two-sided' lifting property against two-sided (co)discrete (co)fibrations.

In order to see from where this lifting propery comes, let us consider its simplest case. Recall from §1.2.1 the inclusions

$$\begin{array}{cccc} \mathbb{1} & & * & \\ & {\scriptstyle i_0}\swarrow & & \searrow{\scriptstyle i_1} \\ \mathbb{2} & 0 & \longrightarrow & 1 \end{array}$$

which satisfied $i_0 \dashv !_{\mathbb{2}} \dashv i_1$, thereby making it a cyclical section. Additionally, let the span

$$\begin{array}{ccc} & E & \\ {\scriptstyle q}\swarrow & & \searrow{\scriptstyle p} \\ A & & B \end{array}$$

be a two-sided discrete fibration. Given an arrow $g : d \to b$, and an object $i \in E$ such that $p(i) = b$, there is a unique lift $\breve{g}$ above g

that is vertical w.r.t. to q:

$$
\begin{array}{lccl}
 & d' \overset{\breve{g}}{\dashrightarrow} i & & \\
\text{over} & d \xrightarrow{g} b & & \text{in } B \\
\text{and} & q(i) = q(i) & & \text{in } A
\end{array}
$$

Part of this can be captured as a diagonal filler for the diagram

$$
\begin{array}{ccc}
\mathbb{1} & \xrightarrow{i} & E \\
{\scriptstyle i_1}\big\downarrow & \overset{\breve{g}}{\nearrow} & \big\downarrow{\scriptstyle p} \\
\mathbb{2} & \xrightarrow[g]{} & B
\end{array}
$$

Commutation of the diagram describes i as being above the codomain of g, whereas commutation of the lower triangle places the lift $\breve{g}$ above g, and commutation of the upper triangle specifies that the codomain of the lift is i. Similarly, we may find a unique oplift of $f : a \to e \in A$ with $q(i) = a$ that is vertical w.r.t. to p:

$$
\begin{array}{lccl}
 & i \overset{\widehat{f}}{\dashrightarrow} e' & & \\
\text{over} & p(i) = p(i) & & \text{in } B \\
\text{and} & a \xrightarrow{f} e & & \text{in } A
\end{array}
$$

and we can capture this as a diagonal filler in the diagram

$$
\begin{array}{ccc}
\mathbb{1} & \xrightarrow{i} & E \\
{\scriptstyle i_0}\big\downarrow & \overset{\widehat{f}}{\nearrow} & \big\downarrow{\scriptstyle q} \\
\mathbb{2} & \xrightarrow[f]{} & B
\end{array}
$$

We can now place these diagrams side by side:

$$
\begin{array}{lccl}
 & d' \overset{\breve{g}}{\dashrightarrow} i \overset{\widehat{f}}{\dashrightarrow} e' \quad (h : d' \dashrightarrow e') & & \\
\text{over} & d \xrightarrow{g} b = b & & \text{in } B \\
\text{and} & a = a \xrightarrow{f} e & & \text{in } A
\end{array}
$$

Thus, the composite $h \overset{\text{def}}{=} \widehat{f} \circ \breve{g}$ is over both f and g. However, *the object i has been 'lost' after composition.* Put simply, if we lift and oplift 'at the same time,' we lose the 'object control' that was afforded previously by the commutation of the upper triangle (i.e. that the codomain or domain of the lift was indeed the object i). The most we can now say is that the diagrams

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{i} & E \\ {\scriptstyle i_0}\downarrow & \nearrow{\scriptstyle h} & \downarrow{\scriptstyle q} \\ \mathbb{2} & \xrightarrow[f]{} & A \end{array} \qquad \begin{array}{ccc} \mathbb{1} & \xrightarrow{i} & E \\ {\scriptstyle i_1}\downarrow & \nearrow{\scriptstyle h} & \downarrow{\scriptstyle p} \\ \mathbb{2} & \xrightarrow[g]{} & B \end{array}$$

have a common *lower diagonal filler* h. The symbol

marks a region of a 'commutative' diagram that is *punctured*, i.e. a region that does not commute.

We can express this lifting property more concisely with the introduction of some notation.

Definition 28 (Two-sided filling problem)**.**

(1) A *lower lifting problem* in a category $\mathcal{E}$ consists of two commuting diagrams

$$\begin{array}{ccc} D & \xrightarrow{i} & E \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle q} \\ C & \xrightarrow[f]{} & A \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{i} & E \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[g]{} & B \end{array}$$

where $D \xrightarrow{i_0} C \xleftarrow{i_1} D$ and $A \xleftarrow{q} E \xrightarrow{p} B$. A *solution* to this lower lifting problem is a *lower diagonal filler*, i.e. an arrow d that fits in both diagrams, and makes the lower triangles commute:

$$\begin{array}{ccc} D & \xrightarrow{i} & E \\ {\scriptstyle i_0}\downarrow & \nearrow{\scriptstyle d} & \downarrow{\scriptstyle q} \\ C & \xrightarrow[f]{} & A \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{i} & E \\ {\scriptstyle i_1}\downarrow & \nearrow{\scriptstyle d} & \downarrow{\scriptstyle p} \\ C & \xrightarrow[g]{} & B \end{array}$$

If L is a class of 'diagonal' cospans (i.e. with both arms having the same source) for which there exist solutions for any lower lifting problem where the right-hand span comes from a class R of spans, then we write $\mathsf{L} \pitchfork_{LD} \mathsf{R}$.

(2) An *upper lifting problem* in a category $\mathcal{E}$ consists of two commuting diagrams

$$\begin{array}{ccc} B & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle r_0} \\ C & \xrightarrow[h]{} & D \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle r_1} \\ C & \xrightarrow[h]{} & D \end{array}$$

where $B \xrightarrow{j} C \xleftarrow{i} A$ and $D \xleftarrow{r_0} X \xrightarrow{r_1} D$. A *solution* to this upper lifting problem is a *upper diagonal filler*, i.e. an arrow d that fits in both diagrams, and makes the lower triangles commute:

$$\begin{array}{ccc} B & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle r_0} \\ C & \xrightarrow[h]{} & D \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{g} & X \\ {\scriptstyle i}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle r_1} \\ C & \xrightarrow[h]{} & D \end{array}$$

If L is a class of cospans for which there exist solutions for any upper lifting problem where the right-hand span comes from a class R of 'diagonal' spans (i.e. spans whose legs have the same target), then we write $\mathsf{L} \pitchfork_{UD} \mathsf{R}$.

We can now show that there eixst lower diagonal fillers for the structures we discussed in structures we defined in §2.1 and §2.6.

Theorem 11. *Given a cyclical section* $D \xrightarrow{i_0} C \xleftarrow{i_1} D$ *and a two-sided discrete fibration* $A \xleftarrow{q} E \xrightarrow{p} B$ *such that the diagrams*

$$\begin{array}{ccc} D & \xrightarrow{i} & E \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle q} \\ C & \xrightarrow[f]{} & A \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{i} & E \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[g]{} & B \end{array}$$

commute, then there is a lower diagonal filler $d : C \to E$*:*

$$\begin{array}{ccc} D & \xrightarrow{i} & E \\ {\scriptstyle i_0}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle q} \\ C & \xrightarrow[f]{} & A \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{i} & E \\ {\scriptstyle i_1}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle p} \\ C & \xrightarrow[g]{} & B \end{array}$$

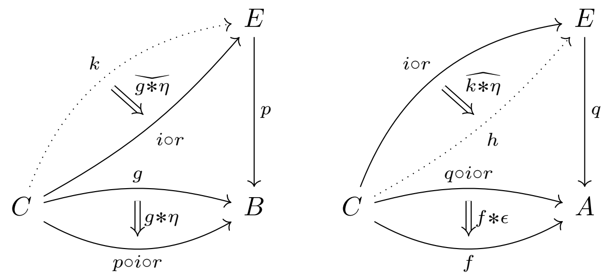
Proof. We have two natural transformations

$$\begin{aligned} \epsilon &: i_0 \circ r \Rightarrow \mathsf{Id}_D \\ \eta &: \mathsf{Id}_D \quad \Rightarrow i_1 \circ r \end{aligned}$$

We whisker them to get

$$\begin{aligned} f * \epsilon &: f \circ i_0 \circ r \Rightarrow f \\ g * \eta &: g \qquad\qquad \Rightarrow g \circ i_1 \circ r \end{aligned}$$

Note that $f \circ i_0 \circ r = q \circ i \circ r$, and $g \circ i_1 \circ r = p \circ i \circ r$. Then, as q and p are respectively a Grothendieck opfibration and a Grothendieck fibration, we construct the following lifts:



We can compose these lifts along their common boundary to obtain

$$\begin{array}{rcccl} k & \overset{\widehat{g\eta}}{\Longrightarrow} & i \circ r & \overset{\breve{f\epsilon}}{\Longrightarrow} & h \\ \text{over } g & \overset{g\eta}{\Longrightarrow} & p \circ i \circ r & = & p \circ i \circ r \\ \text{and } q \circ i \circ r & = & q \circ i \circ r & \overset{f\epsilon}{\Longrightarrow} & f \end{array}$$

By Theorem 9, we can factorise that as

$$\begin{array}{rcccl} k & \overset{\beta}{\Rightarrow} & d \overset{\gamma}{\Rightarrow} & h \\ \text{over } g & = & g \overset{g\eta}{\Longrightarrow} & p \circ i \circ r \\ \text{and } q \circ i \circ r & \overset{f\epsilon}{\Longrightarrow} & f = & f \end{array}$$

so we obtain the desired $d : C \to E$ over f and g. ☐

Remark 6. The careful reader will have noticed that there are a number of things that we assumed and did not eventually use in the preceding proof:

- the fact $r \circ i_0 = \mathsf{Id}$,
- the fact $r \circ i_1 = \mathsf{Id}$,
- the fact $\eta * i_0 = 1$,
- the fact $\epsilon * i_1 = 1$, and
- the uniqueness of the two lifts.

On the one hand, the only elements of the cyclical section that we used were there the natural transformations ϵ and η. The fact that these come with a retraction $r : C \longrightarrow D$ that is common to both ϵ and η was what made the two lifts composable.

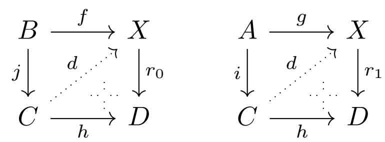
On the other hand, we only used one thing that is special to two-sided discrete fibrations, namely the fact that any morphism in the total category E can be written as a strict composite of (op)lifts of the 'projections' under q and p. In a more general *two-sided fibration* (Loregian and Riehl, 2018) it is not in general the case that those (op)lifts are strictly composable, as isomorphisms may be required to mediate between the two.

We can also prove the following theorem in a completely dual manner.

Theorem 12. *Given a two-sided codiscrete cofibration* $B \xrightarrow{j} C \xleftarrow{i} A$ *and a cyclical retraction* $D \xleftarrow{r_0} X \xrightarrow{r_1} D$ *such that the diagrams*

$$\begin{array}{ccc} B & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle r_0} \\ C & \xrightarrow[h]{} & D \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle r_1} \\ C & \xrightarrow[h]{} & D \end{array}$$

commute, then there is an upper diagonal filler $d : C \to X$*:*



3. Meiotopy Structures

Now let $\mathcal{E}$ be a finitely complete and cocomplete category.

Definition 29 (Meiotopy structure)**.** A *meiotopy structure* on $\mathcal{E}$ consists of eight classes of maps, namely

(1) the *fibrations*, denoted by $\mathcal{F}$,
(2) the *cofibrations*, denoted by $\mathcal{C}$,
(3) the *future sections*, denoted by $\mathcal{FS}$,
(4) the *future retractions*, denoted by $\mathcal{FR}$,

as well as their 'op duals,' namely

(5) the *opfibrations*, denoted by $\mathcal{F}^{\mathrm{op}}$,
(6) the *opcofibrations*, denoted by $\mathcal{C}^{\mathrm{op}}$,
(7) the *past sections*, denoted by $\mathcal{PS}$,
(8) the *past retractions*, denoted by $\mathcal{PR}$.

as well as two classes of spans,

(1) the *cyclical sections*, denoted by $\mathcal{CS}$,
(2) the *two-sided fibrations*, denoted by $\mathcal{TSF}$,

and two classes of cospans,

(1) the *cyclical retractions*, denoted by $\mathcal{CR}$,
(2) the *two-sided cofibrations*, denoted by $\mathcal{TSC}$.

These classes satisfy the following properties:

(1) Sections are left *right-twist-closed*: letting $\mathbb{X} \in \{\mathcal{F}, \mathcal{P}\}$, if $g \circ f \in \mathbb{X}\mathcal{S}$ and $g \in \mathcal{FS} \cup \mathcal{PS}$, then $f \in \mathbb{X}\mathcal{S}$.
(2) Retractions are left *left-twist-closed*: letting $\mathbb{X} \in \{\mathcal{F}, \mathcal{P}\}$, if $g \circ f \in \mathbb{X}\mathcal{R}$ and $f \in \mathcal{FR} \cup \mathcal{PR}$, then $g \in \mathbb{X}\mathcal{R}$.
(3) $(\mathcal{FS}, \mathcal{F})$, $(\mathcal{C}, \mathcal{FR})$, $(\mathcal{PS}, \mathcal{F}^{\mathrm{op}})$, $(\mathcal{C}^{\mathrm{op}}, \mathcal{PR})$ are weak factorisation systems.
(4) $\mathcal{CS} \pitchfork_{LD} \mathcal{TSF}$
(5) $\mathcal{TSC} \pitchfork_{UD} \mathcal{CR}$

By Prop. 4, we automatically have that

- all eight classes of morphisms are closed under composition;
- $\mathcal{FS}$, $\mathcal{C}$, $\mathcal{PS}$, and $\mathcal{C}^{\mathrm{op}}$ are closed under pushouts, coproducts, and retracts;
- $\mathcal{F}$, $\mathcal{FR}$, $\mathcal{F}^{\mathrm{op}}$, and $\mathcal{PR}$ are closed under pullbacks, products, and retracts; and
- identity morphisms have the LLP and the RLP with respect to any morphism, so they are in all eight classes.

3.1. **Fibrance, Cofibrance, Retractibility, Sectionality.**

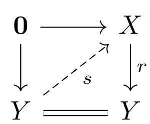
Definition 30.

(1) An object X is *(op)fibrant* when the unique arrow $X \to \mathbf{1} \in \mathcal{F}$ ($\in \mathcal{F}^{\mathrm{op}}$).
(2) An object A is *(op)cofibrant* when the unique arrow $\mathbf{0} \to A \in \mathcal{C}$ ($\in \mathcal{C}^{\mathrm{op}}$).
(3) An object A is *(op)sectional* when the unique arrow $\mathbf{0} \to A \in \mathcal{FS}$ ($\in \mathcal{PS}$).
(4) An object X is *(op)retractible* when the unique arrow $X \to \mathbf{1} \in \mathcal{FR}$ ($\in \mathcal{PR}$).

Proposition 24.

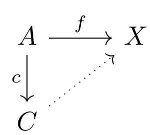
(1) *Any (past) future retraction* to *a (op)cofibrant object is a retraction.*
(2) *Any (past) future section* from *a (op)fibrant object is a section.*
(3) *Any (op)fibration* to *a (op)sectional object is a retraction.*
(4) *Any (op)cofibration* from *a (op)retractible object is a section.*

Proof. If $r : X \to Y$ is a future retraction and Y is cofibrant, we have

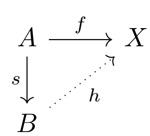


and s is the requisite section. The rest are similar. □

It is worth thinking about the above definitions for a moment. If X is *retractible* (i.e. intuitively has a 'centre of retraction,' a point to which the whole object can be directionally collapsed), then one can fill along any cofibration $c \in \mathcal{C}$:



Similarly, if X is *fibrant*, then one can fill along any future section:



Intuitively, h first retracts B to A, and does whatever f would to A.

3.2. **Pipes, paths, and meiotopy.**

Definition 31.

(1) A *pipe* for $B \in \mathcal{E}$ is an object $\downarrow B \in \mathcal{E}$ along with a factorisation

$$B + B \xrightarrow{[i_0,i_1]} \downarrow B \xrightarrow{r} B$$

of the codiagonal map $\nabla_B : B + B \to B$, where $[i_0, i_1]$ is a a cospan in $\mathcal{CS}$, i.e. a cyclical section.

(2) A *path object* for $X \in \mathcal{E}$ is an object $X^{\downarrow} \in \mathcal{E}$ along with a factorisation

$$X \xrightarrow{s} X^{\downarrow} \xrightarrow{\langle p_1,p_0 \rangle} X \times X$$

of the diagonal map $\Delta_X : X \to X \times X$, where $\langle p_1, p_0 \rangle$ is a span in $\mathcal{CR}$, i.e. a cyclical retraction.

Definition 32 (Meiotopy)**.**

(1) A *left meiotopy* from $f : X \to Y$ to $g : X \to Y$ is a morphism

$$H : \downarrow X \to Y$$

for some pipe $X + X \xrightarrow{[i_0,i_1]} \downarrow X \xrightarrow{r}$ for X, such that

$$H \circ i_0 = f \quad \text{and} \quad H \circ i_1 = g$$

If there exists such a left meiotopy, we write $f \rightsquigarrow_\ell g$.

(2) A *right meiotopy* from $f : X \to Y$ to $g : X \to Y$ is a morphism

$$K : X \to Y^{\downarrow}$$

for some path object $Y \xrightarrow{s} Y^{\downarrow} \xrightarrow{\langle p_1,p_0 \rangle} Y$ for Y, such that

$$p_0 \circ K = f \quad \text{and} \quad p_1 \circ K = g$$

If there exists such a right meiotopy, we write $f \rightsquigarrow_r g$.

Definition 33 (Very good)**.**

(1) A pipe object $B + B \xrightarrow{[i_0,i_1]} \downarrow B \xrightarrow{r} B$ is *very good* if $[i_0, i_1] \in \mathcal{TSC}$, i.e. $[i_0, i_1]$ is a two-sided cofibration.

(2) A path object $X \xrightarrow{s} X^{\downarrow} \xrightarrow{\langle p_1,p_0 \rangle} X \times X$ is *very good* if $\langle p_0, p_1 \rangle \in \mathcal{TSF}$, i.e. $\langle p_0, p_1 \rangle$ is a two-sided fibration.

Proposition 25.

(1) *If* $f \rightsquigarrow_\ell g : B \to X$ *and* $h : X \to Y$*, then* $h \circ f \rightsquigarrow_\ell h \circ g$.
(2) *The relation* $\rightsquigarrow_\ell$ *on* $\mathcal{E}(B, X)$ *is reflexive and transitive.*
(3) *If* $f \rightsquigarrow_\ell g : B \to X$*, then* $f \rightsquigarrow_r g : B \to X$.

Proof.

(1) Trivial.

(2) Let $\downarrow B$ be a pipe for B. Then $\downarrow B \xrightarrow{r} B \xrightarrow{f} Y$ witnesses $f \rightsquigarrow_\ell f$.

In order to prove transitivity,[6] suppose we have pipes

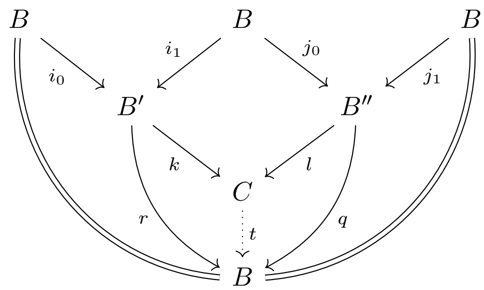
$$B + B \xrightarrow{[i_0,i_1]} B' \xrightarrow{r} B, \qquad B + B \xrightarrow{[j_0,j_1]} B'' \xrightarrow{q} B$$

along with left meiotopies

$$H' : B' \to X, \qquad H'' : B'' \to X$$

[6]In model categories this step would require B to be cofibrant; here this mysteriously vanishes.

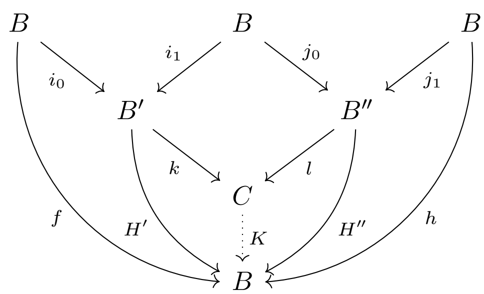
that witness $f \rightsquigarrow_\ell g$ and $g \rightsquigarrow_\ell h$ respectively. We stick together the two pipes along their common boundary (which corresponds to g) by taking the cospan pushout, namely



By the axioms, the cospan $[k \circ i_0, l \circ j_1]$ is a cyclical section. Moreover, $r \circ i_1 = q \circ j_0 = id$, hence there is a unique morphism $t : C \to B$ that makes the two triangles commute, by the universal property of the pushout. Then

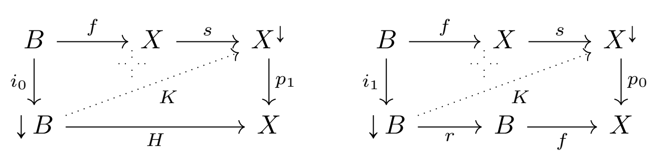
$$B + B \xrightarrow{[k \circ i_0, l \circ j_1]} C \xrightarrow{t} B$$

is a factorisation of the codiagonal, and hence C is a pipe object for B. It remains to construct the meiotopy; we define it to be K in



Then K witnesses $f \rightsquigarrow_\ell h$ with respect to $[k \circ i_0, l \circ j_1]$.

(3) Suppose $H : \downarrow B \to X$ witnesses $f \rightsquigarrow_\ell g$, where $B + B \xrightarrow{[i_0, i_1]} \downarrow B \xrightarrow{r} B$ is a pipe for B. Let $X \xrightarrow{s} X^{\downarrow} \xrightarrow{\langle p_1, p_0 \rangle} X \times X$ be a *very good* path object for X. Then $[i_0, i_1] \in \mathcal{CS}$, $\langle p_1, p_0 \rangle \in \mathcal{TSF}$, and hence we can find a lower diagonal filler in



It follows that $K \circ i_0 : B \to X^{\downarrow}$ witnesses $f \rightsquigarrow_r g$, as

$$p_0 \circ K \circ i_0 = f \circ r \circ i_0 = f$$
$$p_1 \circ K \circ i_0 = H \circ i_0 = g$$

□

In a wholly dual manner,

Proposition 26.

(1) *If* $f \rightsquigarrow_r g : B \to X$ *and* $h : A \to A$*, then* $f \circ h \rightsquigarrow_r g \circ h$*.*
(2) *The relation* $\rightsquigarrow_r$ *on* $\mathcal{E}(B, X)$ *is reflexive and transitive.*
(3) *If* $f \rightsquigarrow_r g : B \to X$*, then* $f \rightsquigarrow_\ell g : B \to X$*.*

Hence, the relations $\rightsquigarrow_\ell$ and $\rightsquigarrow_r$, which are in fact the same relation, are *preorders*, and they are *compatible with composition*.

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