A Type-Theoretic Alternative to LISP

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The Semantics of Intensionality

- I have been trying to understand the logical and computational phenomenon of *intensionality*.
- Intensionality occurs when *mathematical objects can be seen in two ways*:
  - extensionally, i.e. *abstractly*, up to extensional equality
  - intensionally, through *their descriptions*, or syntax
- Extremely common in logic and computer science (but occurs elsewhere too).
- Some tools:
  - proof theory
  - Curry-Howard correspondence
  - modal logic + modal λ-calculus
  - category theory
  - some aspects of computability + realizability

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The Curry-Howard Correspondence

$$\text{formulae} = \text{types}$$
$$\text{proofs (of natural deduction)} = \text{programs}$$
$$\text{reduction/simplification} = \text{computation}$$

$$\frac{\dfrac{\vdots}{\Gamma \vdash A} \quad \dfrac{\vdots}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B} \implies \frac{\dfrac{\vdots}{\Gamma \vdash M : A} \quad \dfrac{\vdots}{\Gamma \vdash N : B}}{\Gamma \vdash \langle M, N \rangle : A \times B}$$

$$\frac{\dfrac{\vdots}{\Gamma \vdash A \wedge B}}{\Gamma \vdash A} \implies \frac{\dfrac{\vdots}{\Gamma \vdash P : A \times B}}{\Gamma \vdash \pi_1(P) : A}$$

The Curry-Howard Correspondence

$$
\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma \vdash A} \quad \dfrac{\vdots}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}}{\Gamma \vdash A} \quad \longrightarrow \quad \dfrac{\mathcal{D}}{\Gamma \vdash A}
$$

We write this as a reduction of proof terms:

$$\pi_1(\langle M, N \rangle) \longrightarrow M$$

Intensionality

- Intensionality is difficult: the waters are treacherous.
- Famous theorems of Gödel, Tarski and Rice: **interplay between intension (= Gödel number, index of computable function) and extension (= formula, computable function)**.
- In general, *one cannot go from extension to intension*:
  - cannot get an index for a computable function from the function itself (**within the computer**);
  - we cannot obtain the Gödel number of a logical formula (**within the logic**); and so on...
- An example from untyped λ-calculus [Barendregt, 1991]:
  Suppose $\mathbf{Q} \in \Lambda$ such that $\mathbf{Q}\ M =_\beta \ulcorner M \urcorner$. Then

$$\ulcorner M \urcorner =_\beta \mathbf{Q}\ M =_\beta \mathbf{Q}\ (\mathbf{I}\ M) =_\beta \ulcorner \mathbf{I}\ M \urcorner$$

which makes two distinct nf's equal.
But the λ-calculus is confluent, hence consistent.

But...

Once something is already 'quoted' (= intension) everything is OK.
E.g. there are $\mathbf{gnum}, \mathbf{app} \in \Lambda$ such that

$$\mathbf{gnum} \ulcorner M \urcorner =_\beta \ulcorner\ulcorner M \urcorner\urcorner \quad \text{and} \quad \mathbf{app} \ulcorner M \urcorner \ulcorner N \urcorner =_\beta \ulcorner M\, N \urcorner$$

Also, Kleene proved:

$$\exists \mathbf{E} \in \Lambda^0.\ \forall M \in \Lambda^0.\ \mathbf{E} \ulcorner M \urcorner =_\beta M$$

So

- intensional operations are admissible
- can go from *intension* ($\ulcorner M \urcorner$) to *extension* (M)
- but *not the other way*.

Idea

Use **types** to stop the flow of information!

To understand intensionality, use types!

Strangely, intensionality follows a typing discipline.
If $M : A$, then let $\ulcorner M \urcorner : \Box A$.

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If $M : A$, then let $\ulcorner M \urcorner : \Box A$.

- For **gnum** $\ulcorner M \urcorner =_\beta \ulcorner\ulcorner M \urcorner\urcorner$, we need

$$\mathbf{gnum} : \Box A \to \Box\Box A$$

- For **app** $\ulcorner M \urcorner \ulcorner N \urcorner =_\beta \ulcorner M\, N \urcorner$, we need

$$\mathbf{app} : \Box(A \to B) \to \Box A \to \Box B$$

- For **E** $\ulcorner M \urcorner =_\beta M$, we need

$$\mathbf{E} : \Box A \to A$$

This is the modal logic S4!
First noticed by Neil Jones (1980s); used by Davies and Pfenning for homogeneous metaprogramming (mid-1990s).

Intensional Recursion

A strange phenomenon that is not well-understood.

Theorem (First Recursion Theorem)

$\forall f \in \Lambda.\ \exists u \in \Lambda.\ u = f\ u$

Theorem (Second Recursion Theorem)

$\forall f \in \Lambda.\ \exists u \in \Lambda.\ u = f \ulcorner u \urcorner$

In the second, u has access to its own code.
The second implies the first (use the interpreter).

Question

What is the programming power of this strong type of recursion?

LISP is untyped, unstructured, unhelpful for understanding this.

And the type of intensional recursion is...?

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Hence the logical meaning of the SRT:

Logical interpretation of the Second Recursion Theorem

From $f : \Box A \to A$, we obtain $u : A$ such that

$$u = f \ulcorner u \urcorner$$

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Logical interpretation of the Second Recursion Theorem

From $f : \Box A \to A$, we obtain $u : A$ such that

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It's Löb's rule from provability logic!

$$\frac{\Box A \to A}{A}$$

Davies-Pfenning S4: a dual-context system

- Judgments: $\Delta\,;\Gamma \vdash M : A$ where $\begin{cases} \Delta = \text{modal/code variables} \\ \Gamma = \text{intuitionistic/value variables} \end{cases}$

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- Using modal variables (internalises $\Box A \to A$):

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- $\Box$ is introduced if and only if *all variables are modal*:

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(Internalises 4 and K/normality.)

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  (Internalises 4 and K/normality.)
- But where are the intensional operations?
  What about intensional recursion?

iPCF: Intensional functions & Intensional Recursion

Let $f : \mathcal{T}(A) \to \mathcal{T}(B)$ be *any function* from closed terms of type A to closed terms of type B. Include it as a constant $\tilde{f} : \Box A \to \Box B$ such that

$$\Delta \,;\, \Gamma \vdash \tilde{f}(\mathsf{box}\, M) = \mathsf{box}\, f(M) : \Box B$$

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And for intensional recursion, include Löb's rule:

$$\frac{\Delta\ ; z : \Box A \vdash M : A}{\Delta\ ; \Gamma \vdash \text{fix } z \text{ in box } M : \Box A}$$

with

$$\text{fix } z \text{ in box } M \longrightarrow \text{box } M[\text{fix } z \text{ in box } M/z]$$

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As long as the congruence rule $\dfrac{\Delta \,;\, \cdot \vdash M = N : A}{\Delta \,;\, \Gamma \vdash \mathsf{box}\ M = \mathsf{box}\ N : \Box A}$ is excluded,

Theorem (K, IMLA 2017)

The above system (Intensional PCF) is confluent, hence consistent.

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Categorical semantics?

We no longer have $\vdash M = N : A$ implying $\vdash \text{box } M = \text{box } N : \Box A$.
C.f. PA $\vdash \phi(x) \leftrightarrow \psi(x) \Longleftarrow \ulcorner \phi(x) \urcorner = \ulcorner \psi(x) \urcorner$, but not conversely!
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SOLUTION: $\Box$ is an *exposure*.

1. Replace categories with *P-categories*: axioms hold up to a partial equivalence relation (PER, = extensional equality).
2. Replace functors by *exposures*, $Q : \mathcal{C} \looparrowright \mathcal{C}$: 'functors' (not quite!) that *do not respect PERs*, but reflect them instead.

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Theorem (K, FoSSaCS 2017)

There are three natural examples of exposures, coming from (a) Peano arithmetic (Gödel numberings); (b) realizability theory (indices for computable functions); (c) homological algebra/group theory. Moreover, there are simple yet abstract categorical versions of the theorems of Gödel, Tarski, Rice, and Kleene (SRT).

Ideas for possible applications

- Foundation for *typed intensional programming*.
  - Essentially a typed LISP with full intensional recursion.
  - Non-functional operations are now possible, in an orderly way.
  - Of course, we have included 'too many intensional functions.' The question of which of those are computable, and which are the right primitives, are both still open.
  - Resulting language can be used to write, amongst other strange programs, a **computer virus** (!).
- Relationship with metaprogramming still unclear, but certainly very interesting.
- Possible application: recursion in type theory.
- Possible application: higher-order computability.